Reference textbook – Erwin Kreyszig, Advance Engineering Mathematics, Calculus 9th ed., Wiley, 2006

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Mathematics 3

#9 Complex Numbers

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Content of this lecture

- Complex numbers
- Complex plane
- Polar form of complex numbers

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COMPLEX NUMBERS

Complex numbers

- Equations without real solutions, such as

$$x^2 = -1$$

or

$$x^2 - 10x + 40 = 0,$$

were observed early in history and led to the introduction of complex numbers.

Complex numbers

- By definition, a *complex number* z is an ordered pair (x, y) of real number x and y, written

$$z = (x, y)$$

- x is called the *real part* and y the imaginary part of z, written

$$x = Re\, z = Re(z), \quad y = Im\, z = Im(z)$$

Complex numbers

- $(0, 1)$ is called the imaginary unit and is denoted by i,

$$i = (0, 1)$$

and $i \times i = -1$.

- Therefore, by the concept of complex numbers, we can find the solution for

$$x^2 = -1$$

- In practice, complex number $z = (x, y)$ can be written as

$$z = x + iy$$

  - Note: Electrical engineers often write j instead of i.

Equality, addition and multiplication

- By definition, two complex number are *equal* if and only if their real parts are equal and their imaginary parts are equal.

- *Addition* of two complex number $z_1 = (x_1,y_1)$ and $z_2 = (x_2,y_2)$ is defined by

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1+x_2, y_1+ y_2)$$

or

$$z_1 + z_2 = (x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+ y_2)$$

Equality, addition and multiplication

- Multiplication is defined by

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

or

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1 x_2 + x_1 iy_2 + x_2 iy_1 + iy_1 iy_2$$

$$= x_1 x_2 + ix_1 y_2 + ix_2 y_1 - y_1 y_2$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Equality, addition and multiplication

- In particular, these two definitions imply that

$$(x_1, 0) + (x_2, 0) = (x_1+x_2, 0)$$

and

$$(x_1, 0)(x_2, 0) = (x_1x_2, 0)$$

as for real number x_1, x_2.

- Hence the complex numbers *extend* the real number.

Example 1

- Let $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$, find $z_1 + z_2$ and $z_1 z_2$.

Subtraction and division

- Subtraction and division are defined as the inverse operations of addition and multiplication, respectively.
- Thus the *difference* $z = z_1 - z_2$ is the complex number z for which $z_1 = z + z_2$.
- Hence,

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

Subtraction and division

- The *quotient* $z = z_1/z_2$ ($z_2 \neq 0$) is the complex number z for which $z_1 = zz_2$.
- Hence, we obtain

$$x_1 = x_2x - y_2y$$

and

$$y_1 = y_2x + x_2y$$

Subtraction and division

- The solution is

$$z = \frac{z_1}{z_2} = x + iy$$

where

$$x = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}$$

and

$$y = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

Subtraction and division

- The *practical rule* used to get this is by multiplying the numerator and denominator of z_1/z_2 by $x_2 - iy_2$ (the *conjugate* of z_2) and simpliflying

$$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$

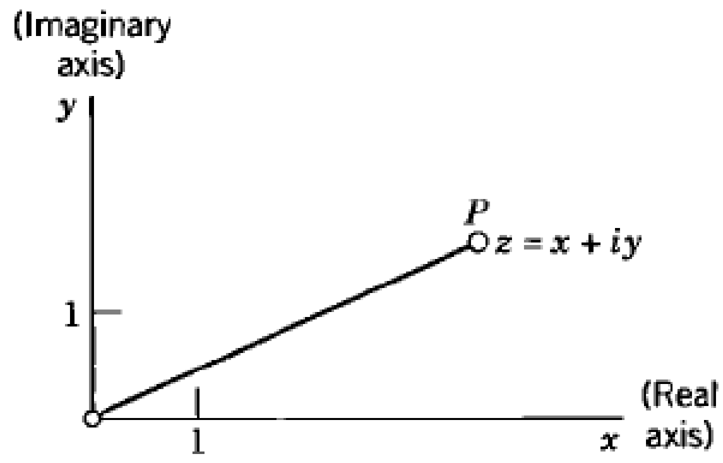
$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

Example 2

- Let $z_1 = 8 + 3i$ and $z_2 = 9 – 2i$, find $z_1 - z_2$ and z_1/z_2.

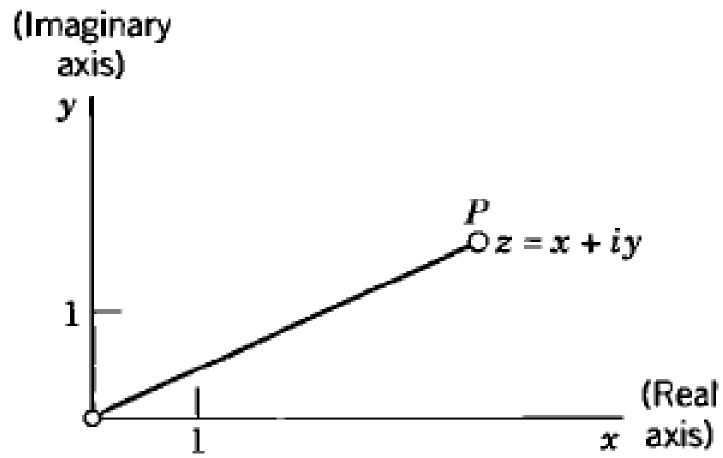
COMPLEX PLANE

Complex plane



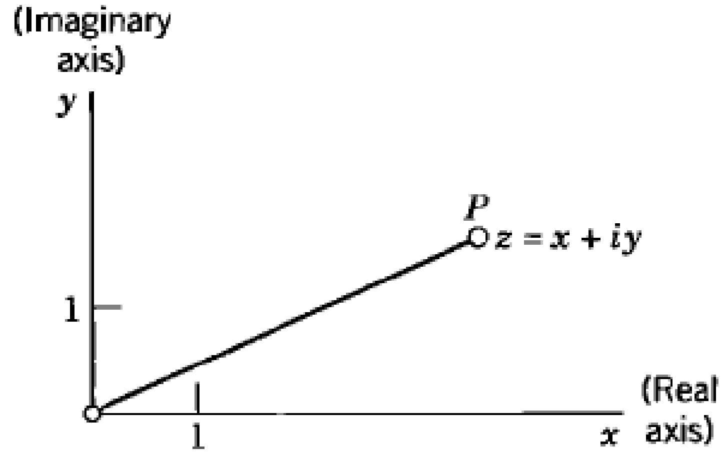
- So far, we have considered the algebra of complex number. Now we will consider the geometrical representation of complex numbers as points in the plane.
- We choose two perpendicular coordinate axes, the horizontal x-axis, called *the real axis*, and the vertical y-axis, called *the imaginary axis*.

Complex plane

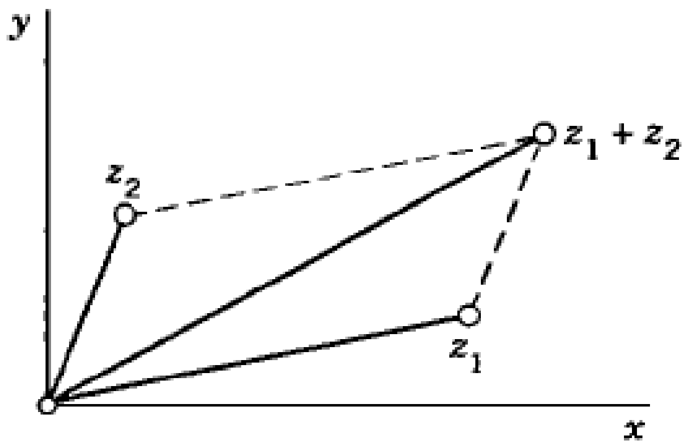


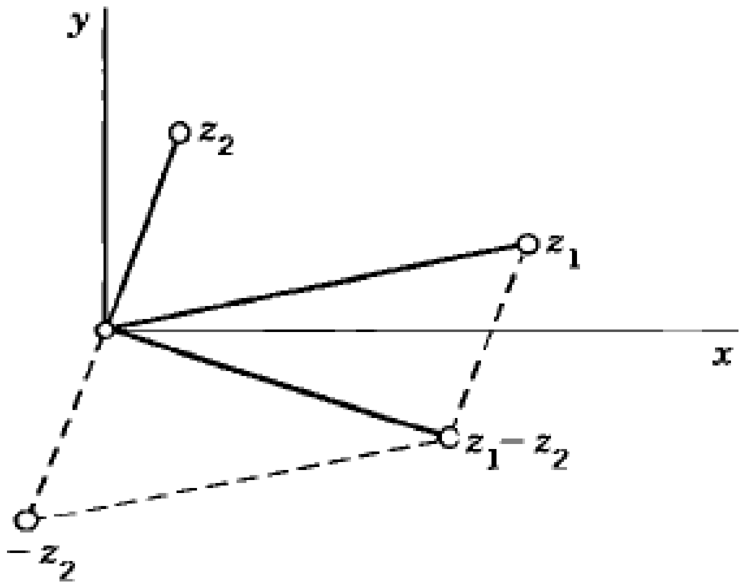
- We now plot a given complex number $z = (x, y) = x + iy$ as the point P with coordinate x, y.
- The xy-plane in which the complex numbers are represented in this way is called the *complex plane*.
  - The complex plane is sometimes called the Argand diagram, after the French mathematician Jean Robert Argand (17868-1822).

Complex plane



- Addition and subtraction can now be visualized as illustrated below:



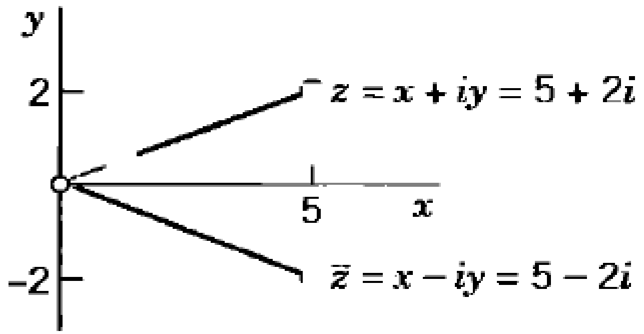


Complex conjugate numbers

- The *complex conjugate* $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy$$

- It is obtained geometrically by reflecting the point z in the real axis.



Complex conjugate numbers

- The complex conjugate is important because it permits us to switch from complex to real.
- Indeed, by multiplication,

$$z\bar{z} = x^2 + y^2$$

- By addition and subtraction,

$$z + \bar{z} = 2x$$

and

$$z - \bar{z} = 2iy$$

Complex conjugate numbers

- Thus we obtain for the real part x and the imaginary part y (not iy !) of $z = x + iy$ by the importan formulas:

$$\operatorname{Re} z = x = \frac{1}{2}(z + \bar{z})$$

$$\operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z})$$

- If z is real, $z = x$, then $z = \bar{z}$ by the definition.

Complex conjugate numbers

- The followings are useful formulas for working with conjugates:

$$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$$

$$\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$$

$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$$

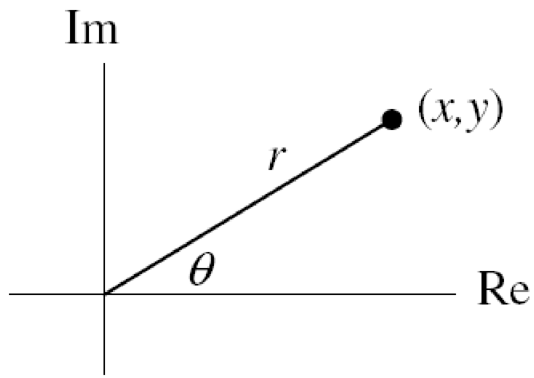
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Example 3

- Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$, then verify that
  - 1) $$\operatorname{Im} z_1 = \frac{1}{2i}(z_1 - \bar{z}_1)$$
  - 2) $$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$$
  - 3) $$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

POLAR FORM OF COMPLEX NUMBERS

Polar form



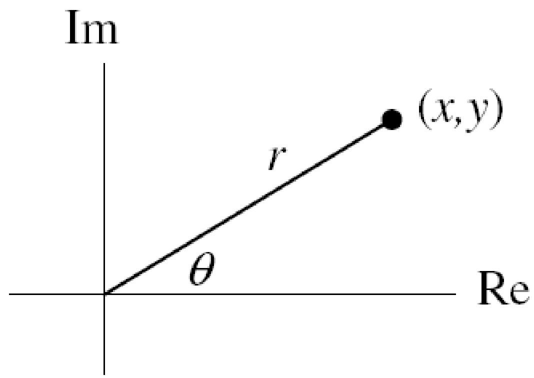
- Besides the *xy*-coordinate, we also employ the usual polar coordinate r, θ defined by

$$x = r\cos\theta,\ y = r\sin\theta$$

- Then $z = x + iy$ takes the so-called *polar form*

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

Polar form

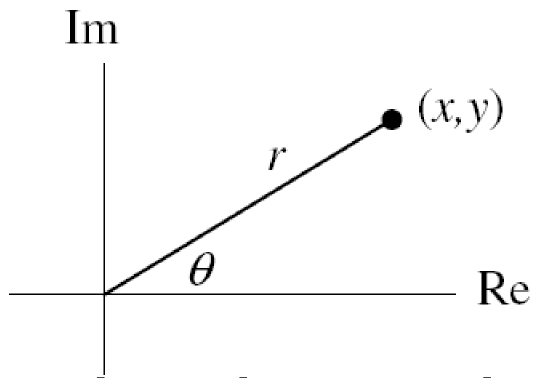


- r is called the *absolute value* or *modulus* of z and is defined by $|z|$. Hence

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

- Geometrically, $|z|$ is the distance of the point z from the origin.
- Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2.

Polar form

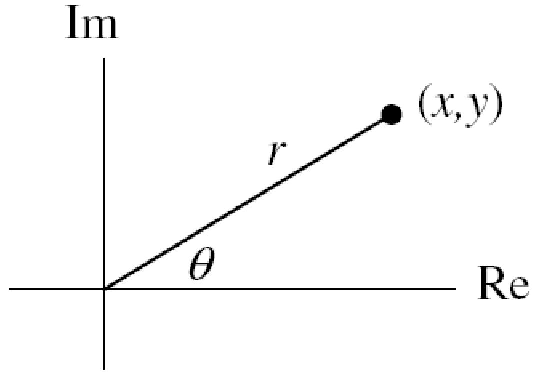


- θ is called the *argument* of z and is denoted by $arg\ z$.
- Thus

$$\theta = \arg z = \arctan \frac{y}{x}$$

- Geometrically, θ is the directed angle from the positive x-axis to the line segment from the origin to the point z.
  - Here, as in calculus, all angles are measured in radians and positive in the counterclockwise sense.

Polar form



- For $z = 0$, this angle θ is undefined.
- For $z \neq 0$, we can have several value of θ that satisfy the equation $\theta = \arg z = \arctan y / x$
- But one often wants to specify a unique value of *arg z* of a given $z \neq 0$.
- For this reason one defines the principle value *Arg z* (with capital *A*) of *arg z* by the double inequality

$$-\pi < \operatorname{Arg} z \leq \pi$$

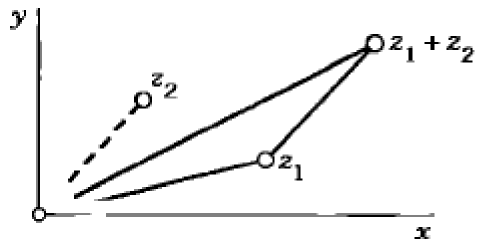
Example

- Find the polar form of $z = 1 + i$.

Triangle inequality

- Inequality such as $x_1 < x_2$ make sense for real numbers, but not in complex because *there is no natural way of ordering complex numbers*.
- However, inequalities between absolute values (which are real), such as $|z_1| < |z_2|$, (meaning that z_1 is closer to origin than z_2) are of great importance.

Triangle inequality



- The daily bread for complex analyst is the *triangle inequality*

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

which we shall use quite frequently.

- By induction, we obtain the generalized triangle inequality

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$$

Example

- If $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then verify that the triangle inequality is true, and sketch the figure.

Multiplicaiton and division in polar form

- Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

  and

  $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

- Then the product of $z_1 z_2$ is

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$
$$+ i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Multiplicaiton and division in polar form

- Taking the absolute for both sides, we see that the absolute value of a product equals the product of the absolute values of the factors,

$$|z_1 z_2| = |z_1||z_2|$$

- The argument of a product equals the sum of the arguments of the factors,

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

Multiplicaiton and division in polar form

- For division, we have $z_1=(z_1/z_2)z_2$.
- Hence

$$|z_1|= |(z_1/z_2)z_2|=|z_1/z_2||z_2|$$

and by the division by $|z_2|$, we have

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Multiplicaiton and division in polar form

- Similary, the argument

$$arg\ z_1 = arg\ [(z_1/z_2)z_2]$$
$$= arg\ (z_1/z_2) + arg\ z_2$$

and by the subtracting by $arg\ z_2$, we have

$$\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2$$

Multiplicaiton and division in polar form

- Combining $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

  and $$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$$

  we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

Multiplicaiton and division in polar form

- From $$|z_1 z_2| = |z_1||z_2|$$
  and
  $$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$
  we obtain by induction for $n = 0, 1, 2, \ldots$
  $$z^n = r^n(\cos n\theta + i \sin n\theta)$$
- Therefore, with $|z| = r = 1$, the formula becomes De Moivre's formula,
  $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$$

Example

- If $z_1 = -2+2i$ and $z_2 = 3i$, then find $z_1 z_2$ and z_1/z_2.