Reference textbook – D.G. Zill and P.D. Shanahan, A First Course in Complex Analysis with Applications, 2nd ed., The Jones and Bartlett Publisher, 2009

13016103 Mathematics 3

#10 Complex Functions

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Content of this lecture

- Powers and Roots
- Complex functions
- Limits and continuity

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POWERS AND ROOTS

Powers and roots

- Recall from algebra that -2 and 2 are said to be squar roots of the number 4 because $(-2)^2 = 4$ and $2^2 = 4$.
- In other words, the two square roots of 4 are distinct solutions of the equation $w^2 = 4$.
- In like manner, we say $w = 3$ is a cube root of 27 since $w^3 = 3^3 = 27$.

Powers and roots

- This last equation points us again in the direction of complex variables since any real number has only *one real* cube root and *two complex* roots.
- In general, we say a number w is an n^{th} root of a nonzero complex number z if $w^n = z$, where n is a positive integer.

Example 1

- Verify that
  $$w_1 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i \text{ and } w_2 = -\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i$$
  are the two square roots of the complex number $z = i$.

Powers and roots

- We will now demonstrate that there are exactly n solutions of the equation $w^n = z$.
- Suppose $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\varphi + i\sin\varphi)$ are polar forms of the complex numbers z and w.
- Then the equation $w^n = z$ becomes

$$\rho^n(\cos n\varphi + i\sin n\varphi) = r(\cos\theta + i\sin\theta).$$

Powers and roots

- We can conclude that

$$\rho^n = r$$

and

$$(\cos n\,\varphi + i\sin n\,\varphi) = (\cos\theta + i\sin\theta).$$

- We define $\rho = \sqrt[n]{r}$ to be the unique positive nth root of the positive real number r.

Powers and roots

- And the definition of equality of two complex numbers implies that

$$\cos n\varphi = \cos\theta \text{ and } \sin n\varphi = \sin\theta.$$

- These equalities indicate that the arguments θ and φ are related by $n\varphi = \theta + 2\pi k$, there k is an integer.
- Thus

$$\varphi = (\theta + 2\pi k)/n\ .$$

Powers and roots

- As k takes on the successive integer value $k = 0, 1, 2, \ldots, n\text{-}1$, we obtain n distinct nth roots of z.
  - Notice that for $k \geq n$, we obtain the same roots because the sine and cosine are 2π-periodic.
- So, suppose k = n + m, where $m = 0, 1, 2, \ldots$, then

$$\varphi = (\theta + 2\pi(n+m))/n = (\theta + 2\pi m)/n + 2\pi$$

Powers and roots

- Therefore

$$\sin\varphi = \sin((\theta + 2\pi m)/n)$$

and

$$\cos\varphi = \cos((\theta + 2\pi m)/n) .$$

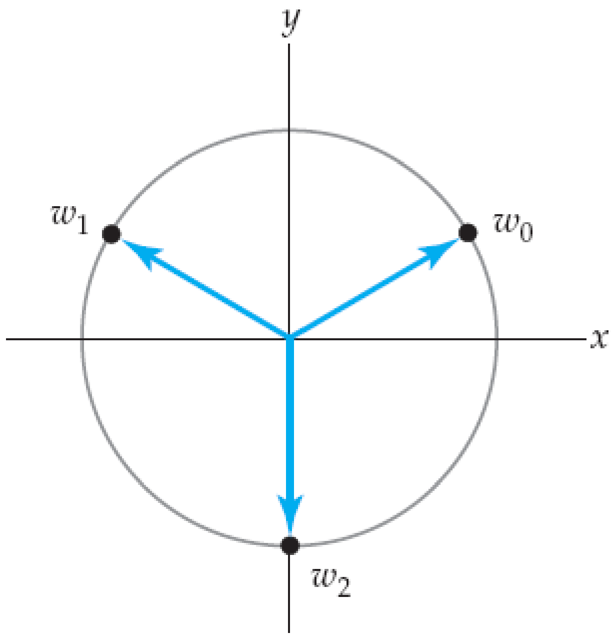
- The n^{th} roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right],$$

where $k = 0, 1, 2, \ldots, n\text{-}1.$

Principle n^{th} roots

- The unique root of a complex number z (obtained by principle value of $arg(z)$ with $k = 0$) is naturally referred to as the principle n^{th} root of w.



Example 2

- Find the three cube root of $z = i$.

  (in other words, find the roots w, where $w^3 = i$)

Example 3

- Find the four roots of $z = 1 + i$.

COMPLEX FUNCTIONS

Functions

- A function f from a set A to a set B is a rule of correspondence that assigns to each element in A one and only element in B

    - Real-valued functions
    - Vector-valued functions
    - Complex-valued functions

Complex functions

Definition 2.1 Complex Function

A **complex function** is a function f whose domain and range are subsets of the set $\mathbf{C}$ of complex numbers.

- A complex function is also called a *complex-valued function of a complex variable*.
- Example, the expression $z^2 - (2 + i)z$ can be evaluated at any complex number z and always yields a single complex number, and so $f(z) = z^2 - (2+i)z$ defines a complex function.

Complex functions

- Values of f are found by using the arithmatic operations for complex numbers.
- For instance, at the points $z = i$ *and* $z = 1 + i$, we have:

$$f(i) = (i)^2 - (2+i)(i) = -1 - 2i + 1 = -2i$$

and

$$f(1+i) = (1+i)^2 - (2+i)(1+i) = 2i - 1 - 3i$$
$$= -1 - i\,.$$

Complex functions

- The expression $g(z) = z + 2Re(z)$ also defines a complex function.
- Some values of g are:

$$g(i) = i + 2Re(i) = i + 2(0) = i$$

and

$$g(2–3i) = (2–3i) + 2Re(2–3i) = 2–3i + 2(2)$$
$$= 6 – 3i$$

Complex functions

- Note:
  - Because the set R of real numbers is a subset of the set C of the complex numbers, *every real-valued function of a real variable is also a complex function*.
  - We will see soon that real-valued functions of two real variables x and y are also special types of complex functions.
  - These functions will play important role in the study of complex analysis.

Real & imaginary parts

- It is often helpful to express the inputs and the outputs of a complex function terms of their real and imaginary parts.
- If $w = f(z)$ is a complex function, then the image of a complex number $z = x + iy$ under f is a complex number $w = u + iv$.
- By simplifying the expression $f(x + iy)$, we can write the real variables u and v in terms of the real variables x and y.

Real & imaginary parts

- For example, by replacing the symbol z with $x + iy$ in the complex function $w = z^2$, we obtain:

$$w = u + iv = (x + iy)^2 = x^2 - y^2 + 2xyi \, .$$

- Therefore, the real variables u and v are given by $u = x^2 - y^2$ and $v = 2xy$, respectively.
- Then both u and v are real functions of the two real variables x and y.

Real & imaginary parts

- That is, by setting $z = x + iy$, we can express any complex function $w = f(z)$ in terms of two real functions as:

$$f(z) = u(x, y) + iv(x, y) .$$

- The functions $u(x, y)$ and $v(x, y)$ are called the *real* and *imaginary parts of f,* respectively.

Example

- Find the real and imaginary parts of the functions:
  - a) $f(z) = z^2 - (2+i)z$
  - b) $g(z) = z + 2Re(z)$

Exponential functions

Definition 2.2 Complex Exponential Function

The function e^z defined by:

$$e^z = e^x \cos y + ie^x \sin y \tag{3}$$

is called the **complex exponential function.**

- By definition, the real and imaginary parts of the complex exponential function are $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$, respectively.

Example 1

- Find the values of the complex exponential funciton ez at the following points:
  - a) $z = 0$
  - b) $z = i$
  - c) $z = 2 + \pi i$

Exponential form of a complex number

- The complex exponential function enables us to express the polar form of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ in particularly convenient and compact form:

$$z = re^{i\theta} .$$

- We call this the exponential form of the complex number z.

Exponential form of a complex number

- For example, a polar form of the complex number $3i$ is $3[cos(\pi/2) + isin(\pi/2)]$, whereas an exponential form of $3i$ is $3e^{i\pi/2}$.
- We call this the exponential form of the complex number z.
  - Note: the complex exponential function is periodic:

$$e^{z+2\pi i} = e^{z}$$

This implies that the complex exponential function has a pure imaginary period $2\pi i$.

Exponential form of a complex number

- Common properties of complex exponential function:

$$e^{0} = 1$$

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

$$(e^{z_1})^{n} = e^{n z_1} \quad for\ n = 0, \pm 1, \pm 2, \ldots$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Example 2

- Verify that $\sqrt{2}e^{i\pi/4}$, $\sqrt{2}e^{i9\pi/4}$, and $\sqrt{2}e^{i17\pi/4}$ are all valid exponential forms of the complex number $1 + i$.

Example 3

- Find the exponential form of $z = x + 0i$.

Summary: the forms of a complex number

- Ordinary form

$$z = (x, y) = x + iy$$

- Polar form

$$z = r(\cos\theta + i\sin\theta)$$

$$r = |z| = (x^2+y^2)^{1/2}, \quad \theta = \arctan(y/x)$$

- Exponential form

$$z = re^{i\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

LIMITS AND CONTINUITY

Limits

- The most important concept in elementary calculus is that of the limit.
- *Complex limits* play an important role in study of complex analysis.
- The concept of a complex limit is similar to that of a real limit in the sense that

$$\lim_{z->z_0} f(z) = L$$

will mean that the values $f(z)$ of the complex function f can be made arbitrarily close the complex number L if values of z are chosen sufficiently close to, but not equal to, the complex number z_0.

Limits

- In a *real* limit, *there are two directions* from which a real variable x can approach to a real value x_0 on the real line, namely, from the left or from the right.
- In a *complex* limit, however, *there are infinitely many directions* from which z can approach z_0 in the complex plane.
- In order for a complex limit to exist, *each way* in which z can approach z_0 *must yield the same limiting value*.

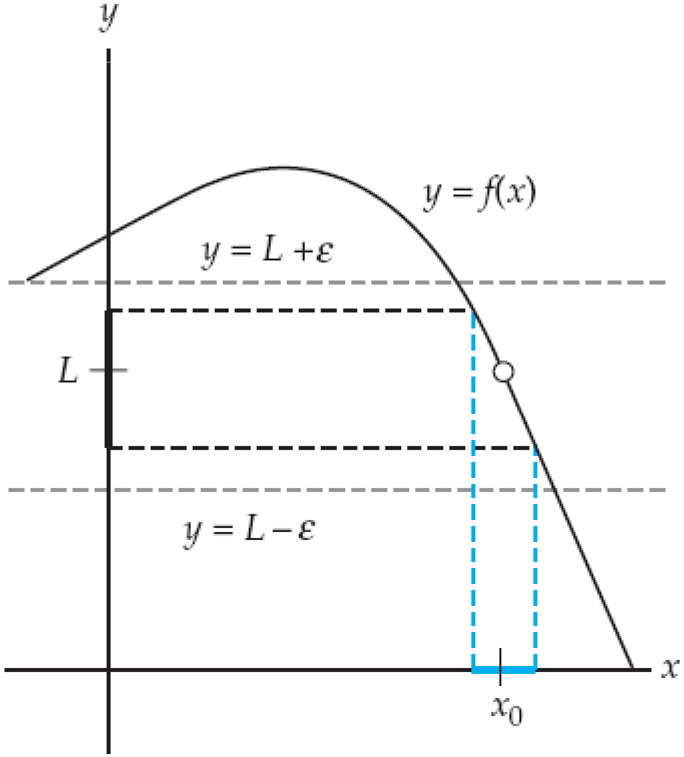
Limits

Limit of a Real Function $f(x)$

The limit of f as x tends x_0 exists and is equal to L

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ (1)

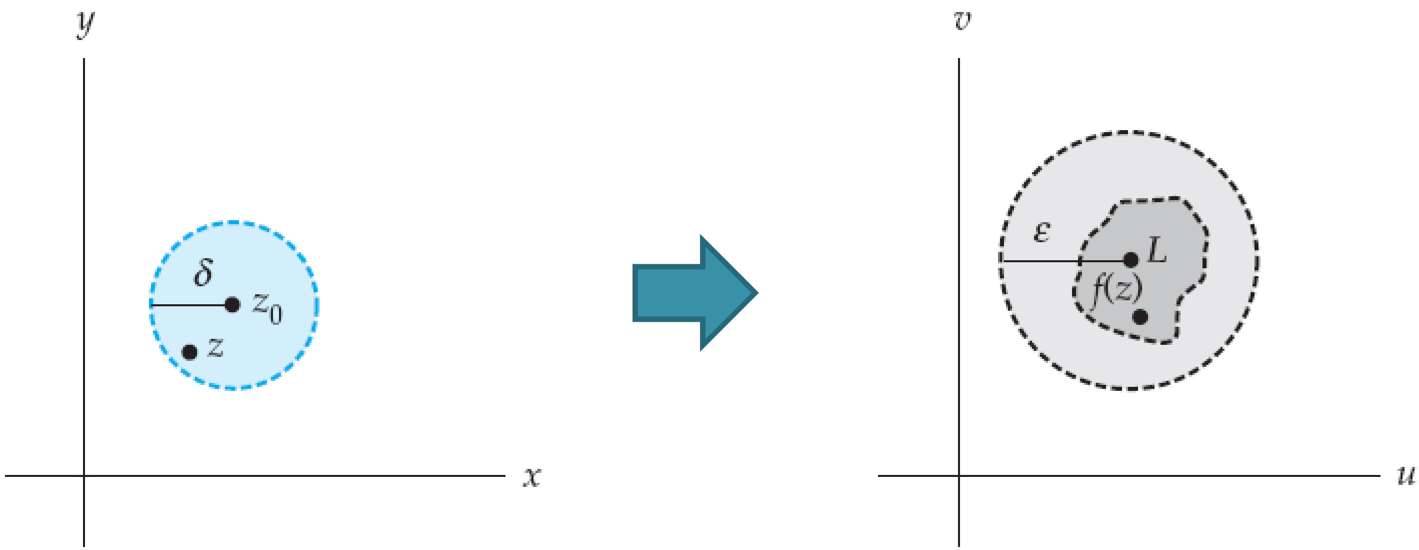
whenever $0 < |x - x_0| < \delta$.



Limits

Definition 2.8 Limit of a Complex Function

Suppose that a complex function f is defined in a deleted neighborhood of z_0 and suppose that L is a complex number. The **limit of f as z tends to z_0 exists and is equal to L**, written as $\lim_{z \to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.



(a) Deleted δ-neighborhood of z_0 **(b)** ε-neighborhood of L

Limits

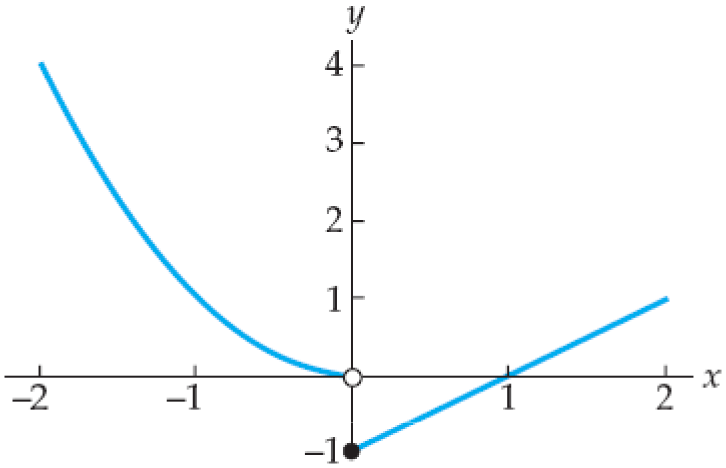


Figure 2.52 The limit of f does not exist as x approaches 0.

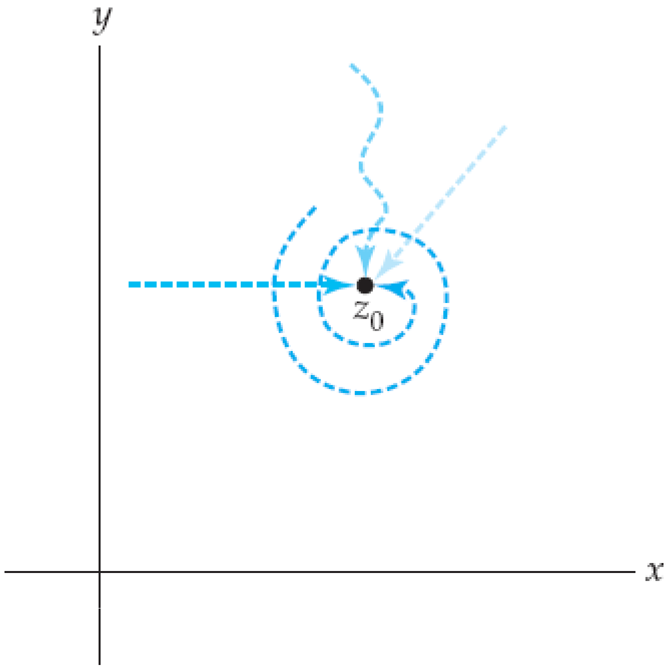


Figure 2.53 Different ways to approach z_0 in a limit

Criterion for the Nonexistence of a Limit

If f approaches two complex numbers $L_1 \neq L_2$ for two different curves or paths through z_0, then $\lim_{z \to z_0} f(z)$ does not exist.

Example 1

- Show that $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

Example 2

- Prove that $\lim_{z \to 1+i} (2+i)z = 1+3i$.

Real multivariable limits

Limit of the Real Function $F(x, y)$

The limit of F as $(x,\ y)$ tends to $(x_0,\ y_0)$ exists and is equal to the real number L if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|F(x,\ y) - L| < \varepsilon$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$. (7)

Theorem 2.1 Real and Imaginary Parts of a Limit

Suppose that $f(z) = u(x,\ y) + iv(x,\ y)$, $z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$. Then $\lim_{z \to z_0} f(z) = L$ if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,\ y) = u_0 \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0.$$

Example 3

- Use Theorem 2.1 to compute $\lim_{z \to 1+i} (z^2 + i)$.

Properties of complex limits

Theorem 2.2 Properties of Complex Limits

Suppose that f and g are complex functions. If $\lim_{z \to z_0} f(z) = L$ and $\lim_{z \to z_0} g(z) = M$, then

(*i*) $\lim_{z \to z_0} cf(z) = cL$, c a complex constant,

(*ii*) $\lim_{z \to z_0} (f(z) \pm g(z)) = L \pm M$,

(*iii*) $\lim_{z \to z_0} f(z) \cdot g(z) = L \cdot M$, and

(*iv*) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$, provided $M \neq 0$.

Additional properties

$$\lim_{z \to z_0} c = c, \quad c \text{ a complex constant},$$

$$\lim_{z \to z_0} z = z_0.$$

Example 4

- Use Theorem 2.2 to compute the limits
  - a) $\lim_{z \to i} \dfrac{(3+i)z^4 - z^2 + 2z}{z+1}$
  - b) $\lim_{z \to 1+\sqrt{3}i} \dfrac{z^2 - 2z + 4}{z - 1 - \sqrt{3}i}$

Continuity

Continuity of a Real Function f(x)

A function f is continuous at a point x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$. (17)

Definition 2.9 Continuity of a Complex Function

A complex function f is **continuous at a point** z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Criteria for Continuity at a Point

A complex function f is continuous at a point z_0 if each of the following three conditions hold:

(*i*) $\lim_{z \to z_0} f(z)$ *exists,*

(*ii*) *f is defined at z_0, and*

(*iii*) $\lim_{z \to z_0} f(z) = f(z_0)$.

Example 5

- Check the continuity of the function $f(z) = z^2 - iz + 2$ at the point $z_0 = 1 - i$.

Example 6

- Show that $f(z) = z^{1/2}$ is discontinuous at the point $z_0 = -1$.

Continuity

Continuity of a Real Function $F(x, y)$

A function F is continuous at a point $(x_0,\ y_0)$ if

$$\lim_{(x,y)\to(x_0,y_0)} F(x,\ y) = F(x_0,\ y_0). \tag{20}$$

Theorem 2.3 Real and Imaginary Parts of a Continuous Function

Suppose that $f(z) = u(x,\ y) + iv(x,\ y)$ and $z_0 = x_0 + iy_0$. Then the complex function f is continuous at the point z_0 if and only if both real functions u and v are continuous at the point $(x_0,\ y_0)$.

Continuity

Theorem 2.4 Properties of Continuous Functions

If f and g are continuous at the point z_0, then the following functions are continuous at the point z_0:

(i) cf, c a complex constant,

(ii) $f \pm g$,

(iii) $f \cdot g$, and

(iv) $\dfrac{f}{g}$ provided $g(z_0) \neq 0$.

Theorem 2.5 Continuity of Polynomial Functions

Polynomial functions are continuous on the entire complex plane $\mathbf{C}$.

Example 7

- Show that $f(z)=\bar{z}$ is continuous on C.

HW

- 1) Find all solutions of the equation

$$z^4 + 1 = 0$$

- 2) Compute the following complex limits:
  - a) $\lim_{z \to 2i} (z^2 - \bar{z})$
  - b) $\lim_{z \to \pi i} e^z$
- 3) Show that the function *f* is discontinuous at the given point:

$$f(z) = \frac{z^2 + 1}{z + i}, \quad z_0 = -i$$