Reference textbook – D.G. Zill and P.D. Shanahan, A First Course in Complex Analysis with Applications, 2nd ed., The Jones and Bartlett Publisher, 2009

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Mathematics 3

#11 Analytic Functions and Integration in the Complex Plane

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Content of this lecture

- Differentiability
- Analyticity
- Cauchy-Riemann equations
- Integration in the complex plane

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DIFFERENTIABILITY

Differentiability

- Suppose $z = x + iy$ and $z_0 = x_0 + iy_0$;
- then the change in z_0 is the difference

$$\Delta z = z - z_0 \text{ or}$$

$$\Delta z = (x - x_0) + i(y - y_0) = \Delta x + i\,\Delta y$$

- If a complex number $w = f(z)$ is defined at z and z_0, then the corresponding change in function is the difference

$$\Delta w = f(z_0 + \Delta z) - f(z_0).$$

Differentiability

- The derivative of the function *f* is defined in terms of a limit of the difference quotient $\Delta w/\Delta z$ as $\Delta z \rightarrow 0$.

Definition 3.1 Derivative of Complex Function

Suppose the complex function f is defined in a neighborhood of a point z_0. The **derivative** of f at z_0, denoted by $f'(z_0)$, is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{1}$$

provided this limit exists.

Differentiability

- If the limit in *(1)* exists, then the function f is said to be differentiable at z_0.

- Two other symbols denoting the derivative of $w = f(z)$ are w' and dw/dz.

- If the latter notation is used, then the value of a derivative at a specified point z_0 is written $\left.\dfrac{dw}{dz}\right|_{z=z_0}$.

Example 1

- Use the definition *3.1* to find the derivative of $f(z) = z^2 - 5z$.

Rules of differentiation

Differentiation Rules

Constant Rules: $$\frac{d}{dz}c = 0 \text{ and } \frac{d}{dz}cf(z) = cf'(z) \tag{2}$$

Sum Rule: $$\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z) \tag{3}$$

Product Rule: $$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z) \tag{4}$$

Quotient Rule: $$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2} \tag{5}$$

Chain Rule: $$\frac{d}{dz}f(g(z)) = f'(g(z))\, g'(z). \tag{6}$$

Rules of differentiation

- The *power rule* for differentiation of powers of z is also valid:

$$\frac{d}{dz} z^n = nz^{n-1}, \quad n \text{ an integer.} \tag{7}$$

- Combining (6) and (7) gives the *power rule for functions*:

$$\frac{d}{dz} [g(z)]^n = n[g(z)]^{n-1} g'(z), \quad n \text{ an integer.} \tag{8}$$

Example 2

- Differentiate:

a) $f(z) = 3z^4 - 5z^3 + 2z$

b) $f(z) = z^2/(4z + 1)$

c) $f(z) = (iz^2 + 3z)^5$

A function that is nowhere differentiable

- For a complex function f to be differentiable at a point z_0, we know from the preceding lecture that the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

  must exist and equal the same complex number from any direction.
- The limit must exist regardless how Δz approaches 0.

A function that is nowhere differentiable

- This means that in complex analysis, the requirement of differentiability of a function $f(z)$ at a point z_0 is a far greater demand than in real calculus of function $f(x)$ where we can approach a real number x_0 on the number line from only two directions.
- An example of a complex function that is not differentiable is $f(z) = x + 4iy$.

Example 3

- Show that the function $f(z) = x + 4iy$ is not differentiable at any point z.

ANALYTICITY

Analytic functions

- Even though the requirement of differentiability is a stringent demand, there is a class of functions that is of great importance whose members satisfy even more severe requirements.
- These functions are called *analytic functions*.

Analytic functions

Definition 3.2 Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

- A function f is analytic in a domain D if it is analytic at every point in D.
- The phrase "analytic on a domain D" is also used but we shall call a function f that is analytic throughout a domain D *holomorphic* or *regular*.

Analytic functions

Definition 3.2 Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

- Note: analyticity at a point is *not* the same as differentiability at a point!
- Analyticity at a point is a neighborhood property; in other words, analyticity is a property that is defined over an open set.

Analytic functions

Definition 3.2 Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

- As an example, the function $f(z) = |z|^2$ is differentiable at $z = 0$ but is not differentiable anywhere else.
- Even though $f(z) = |z|^2$ is differentiable at $z = 0$, it is not analytic at that point because there exists no neighborhood of $z = 0$ throughout which f is differentiable; hence, $f(z) = |z|^2$ is nowhere analytic!

Example

- Show that the function $f(z) = |z|^2$ is differentiable only at $z = 0$.

Entire functions

- A function that is analytic at every point z in the complex plane is said to be an entire function.
- According to the differentiation rules (2), (3), (5), and (7), we can conclude that *polynomial functions* are differentiable at every point z in the complex plane and *rational functions* are analytic throughout any domain D that contain no points at which the denominator is zero.

Entire functions

Theorem 3.1 Polynomial and Rational Functions

(*i*) A polynomial function $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where n is a nonnegative integer, is an entire function.

(*ii*) A rational function $f(z) = \dfrac{p(z)}{q(z)}$, where p and q are polynomial functions, is analytic in any domain D that contains no point z_0 for which $q(z_0) = 0$.

Analyticity of Sum, Product, and Quotient

The sum $f(z) + g(z)$, *difference* $f(z) - g(z)$, *and product* $f(z)g(z)$ *are analytic. The quotient* $f(z)/g(z)$ *is analytic provided* $g(z) \neq 0$ *in* D.

Differentiability & continuity

Theorem 3.2 Differentiability Implies Continuity

If f is differentiable at a point z_0 in a domain D, then f is continuous at z_0.

- As in real analysis, *if a function f is differentiable* at a point, *the function is necessarily continuous* at the point.
- Of course *the converse is not true*; continuity of a function f at a point does not guarantee that f is differentiable at the point.
- Example, $f(z) = x + 4iy$ is continuous everywhere but is nowhere differentiable.

L'Hopital's rule

- L'Hopitals'rule for computing limits of the indeterminate form *0/0*, carries over to complex analysis.

Theorem 3.3 L'Hôpital's Rule

Suppose f and g are functions that are analytic at a point z_0 and $f(z_0) = 0$, $g(z_0) = 0$, but $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}. \tag{13}$$

Example

- Using L'Hopital's rule to compute

$$\lim_{z \to 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i}$$

CAUCHY-RIEMANN EQUATIONS

Cauchy-Riemann equations

- Here, we sahll learn a test for analyticity of a complex function $f(z) = u(x, y) + iv(x, y)$ that is based on partial derivatives of its real and imaginary parts u and v.

Theorem 3.4 Cauchy-Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \qquad (1)$$

Cauchy-Riemann equations

Proof The derivative of f at z is given by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{2}$$

By writing $f(z) = u(x,\ y) + iv(x,\ y)$ and $\Delta z = \Delta x + i\Delta y$, (2) becomes

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}. \tag{3}$$

Since the limit (2) is assumed to exist, Δz can approach zero from any convenient direction. In particular, if we choose to let $\Delta z \to 0$ along a horizontal line, then $\Delta y = 0$ and $\Delta z = \Delta x$. We can then write (3) as

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y) + i\left[v(x + \Delta x, y) - v(x, y)\right]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}. \end{aligned} \tag{4}$$

Cauchy-Riemann equations

The existence of $f'(z)$ implies that each limit in (4) exists. These limits are the definitions of the first-order partial derivatives with respect to x of u and v, respectively. Hence, we have shown two things: both $\partial u/\partial x$ and $\partial v/\partial x$ exist at the point z, and that the derivative of f is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{5}$$

We now let $\Delta z \to 0$ along a vertical line. With $\Delta x = 0$ and $\Delta z = i\Delta y$, (3) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}. \tag{6}$$

Cauchy-Riemann equations

In this case (6) shows us that $\partial u/\partial y$ and $\partial v/\partial y$ exist at z and that

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{7}$$

By equating the real and imaginary parts of (5) and (7) we obtain the pair of equations in (1). ✎

Criterion for Non-analyticity

If the Cauchy-Riemann equations are not satisfied at every point z in a domain D, then the function $f(z) = u(x, y) + iv(x, y)$ cannot be analytic in D.

Example

- Using Cauchy-Riemann equation to show that $f(z) = x + 4iy$ is not analytic.

Example

- Verify that the polynomial function $f(z) = z^2 + z$ satisfies Cauchy-Riemann equations for all z.

Example

- Show that the complex function $f(z) = 2x^2 + y + i(y^2 - x)$ is not analytic at any point.

Criterion for analyticity

- By themselves, the Cauchy-Riemann equations do not ensure analyticity of a function $f(z) = u(x, y) + iv(x, y)$ at a point $z = x + iy$.
- It is possible for the Cauchy-Riemann equations to be satified at z and yet $f(z)$ may not be differentiable at z, or
- $f(z)$ may be differentiable at z but nowhere else.
- In either case, f is not analytic at z.

Criterion for analyticity

- However, when we add the condition of continuity to u and v and to the four partial derivatives $\partial u/\partial x$,$\partial u/\partial y$, $\partial v/\partial x$ and $\partial v/\partial y$, it can be shown that the Cauchy-Riemann equations are not only necessary but also sufficient to guarantee analyticity of $f(z)$ at z.

Criterion for analyticity

Theorem 3.5 Criterion for Analyticity

Suppose the real functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy-Riemann equations (1) at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

Sufficient Conditions for Differentiability

If the real functions $u(x,y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in some neighborhood of a point z, and if u and v satisfy the Cauchy-Riemann equations (1) at z, then the complex function $f(z) = u(x, y) + iv(x, y)$ is differentiable at z and $f'(z)$ is given by (9).

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \qquad (9)$$

Example

- Use Theorem 3.5 to check the analyticity of

$$f(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Cauchy-Riemann equations in polar coordinate

- Indeed, the form $f(z) = u(r, \theta) + iv(r, \theta)$ is often more convenient to use.
- In polar coordinate, the Cauchy-Riemann equations becomes

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

and

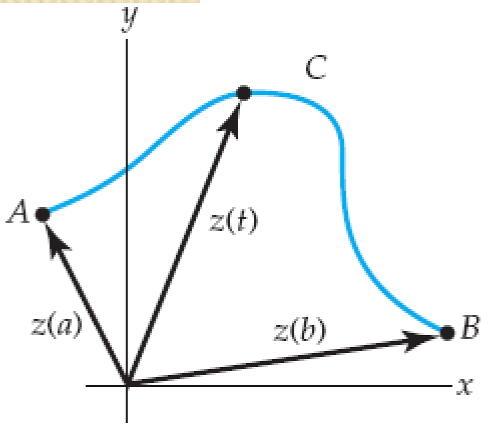
$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{r}e^{-i\theta}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right).$$

INTEGRATION IN THE COMPLEX PLANE

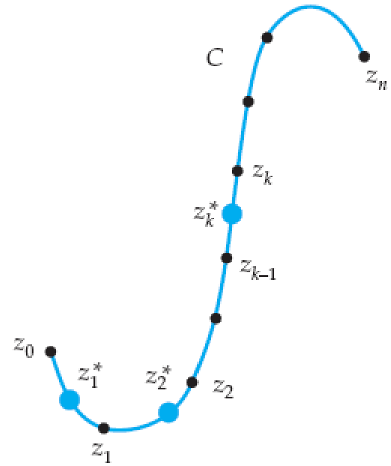
Curve revisited

- Suppose the continuous real-valued functions $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, are parametric equations of a curve C in the complex plane.
- If we use these equations as the real and imaginary parts in $z = x + iy$, we can describe the point z on C by means of a complex-valued function of a real variable t called parametrization of C:

$$z(t) = x(t) + iy(t), \; a \leq t \leq b. \tag{1}$$



Complex integral



- An integral of a function f of a complex variable z that is defined on a contour (piecewise smooth curve) C is denoted by $\int_C f(z)dz$ and is called a *complex integral* or *contour integral*.

Definition 5.3 Complex Integral

The complex integral of f on C is

$$\int_C f(z)\,dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\,\Delta z_k. \qquad (2)$$

Complex integral

- To evaluate a contour integral $\int_C f(z)dz$, let us write (2) in an abbreviated form.

$$\int_C f(z)dz = \lim \sum (u + iv)(\Delta x + i\Delta y)$$
$$= \lim \left[\sum (u\Delta x - v\Delta y) + i \sum (v\Delta x + u\Delta y) \right].$$

- The intepretation is

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i \int_C v\,dx + u\,dy. \qquad (9)$$

Complex integral

- Specifically, the right-hand side becomes

$$\overbrace{\int_a^b [u(x(t),y(t))\ x'(t) - v(x(t),y(t))\ y'(t)]\,dt}^{\int_C u dx - v dy} + i\overbrace{\int_a^b [v(x(t),y(t))\ x'(t) + u(x(t),y(t))\ y'(t)]\,dt}^{\int_C v dx + u dy}. \tag{10}$$

Complex integral

- If we use the complex-valued function (*1*) to describe the contour C, then (*10*) is the same as $\int_a^b f(z(t))z'(t)dt$ when the integrand is

$$f(z(t))\,z'(t) = [u(x(t), y(t)) + iv(x(t), y(t))]\,[x'(t) + iy'(t)]$$

- Thus we arrive at a practical means of evaluating a contour integral.

Theorem 5.1 Evaluation of a Contour Integral

If f is continuous on a smooth curve C given by the parametrization $z(t) = x(t) + iy(t)$, $a \le t \le b$, then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\;z'(t)\,dt. \tag{11}$$

Example

- Evaluate $\int_C \bar{z}\,dz$, where C is given by $x = 3t$, $y = t2$, $-1 \leq t \leq 4$.

Example

- Evaluating $\int_C \bar{z}\,dz$, where C is the circle $x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$.

Properties of contour integral

Theorem 5.2 Properties of Contour Integrals

Suppose the functions f and g are continuous in a domain D, and C is a smooth curve lying entirely in D. Then

(*i*) $\int_C kf(z)\,dz = k\int_C f(z)\,dz$, k a complex constant.

(*ii*) $\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.

(*iii*) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(*iv*) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ denotes the curve having the opposite orientation of C.

Bounding theorem

- Sometimes, it may be useful to find an upper bound for the modulus or absolute value of a contour integral.
- The following theorem is proved by using the form of the triangle inequality.

Theorem 5.3 A Bounding Theorem

If f is continuous on a smooth curve C and if $|f(z)| \leq M$ for all z on C, then $\left|\int_C f(z)\,dz\right| \leq ML$, where L is the length of C.

Example

- Find an upper bound for the absolute value of $\oint_C \frac{e^z}{z+1} dz$ where C is the circle $|z|$ = 4.