Reference textbook – James Stewart, Calculus 6th ed., Thomson, 2009

13016103 Mathematics 3

#1 Parametric Equations

Dr. Ukrit Watchareeruetai, International College, KMITL

Content of this lecture

- Curves
- Curves defined by parametric equations
- Calculus with parametric curves
  - Slope and tangent
  - Area
  - Arc length
  - Surface

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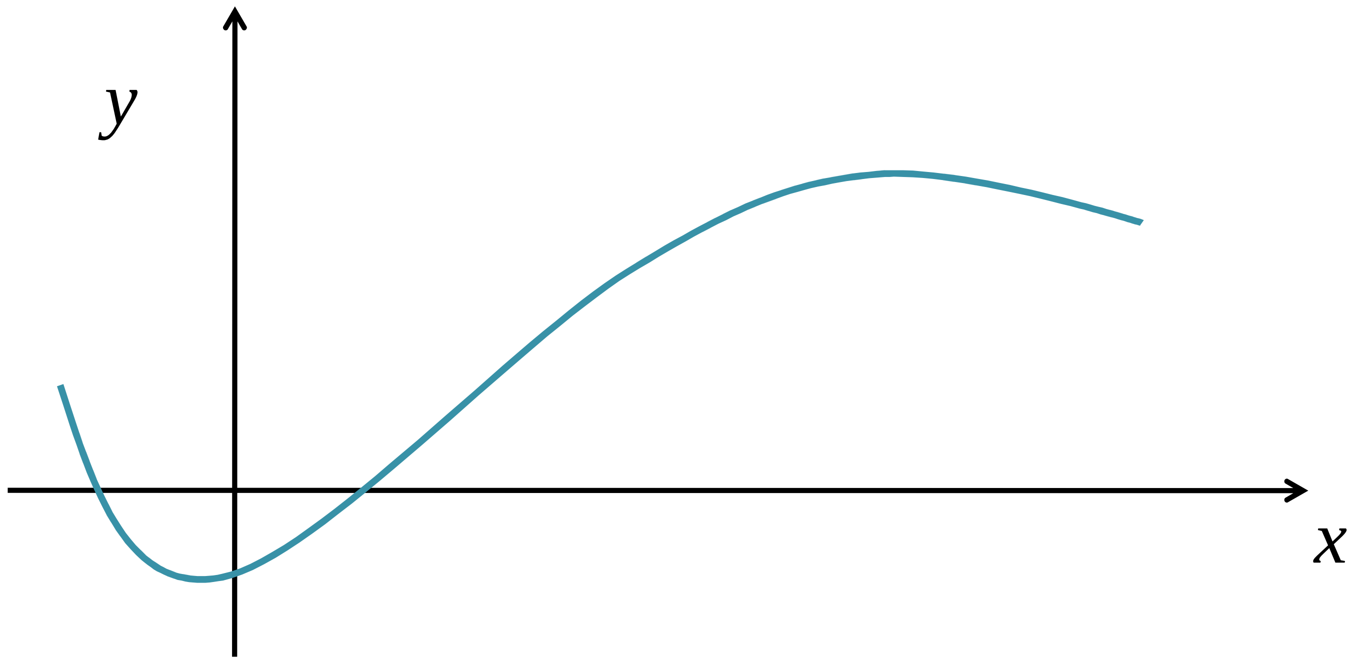
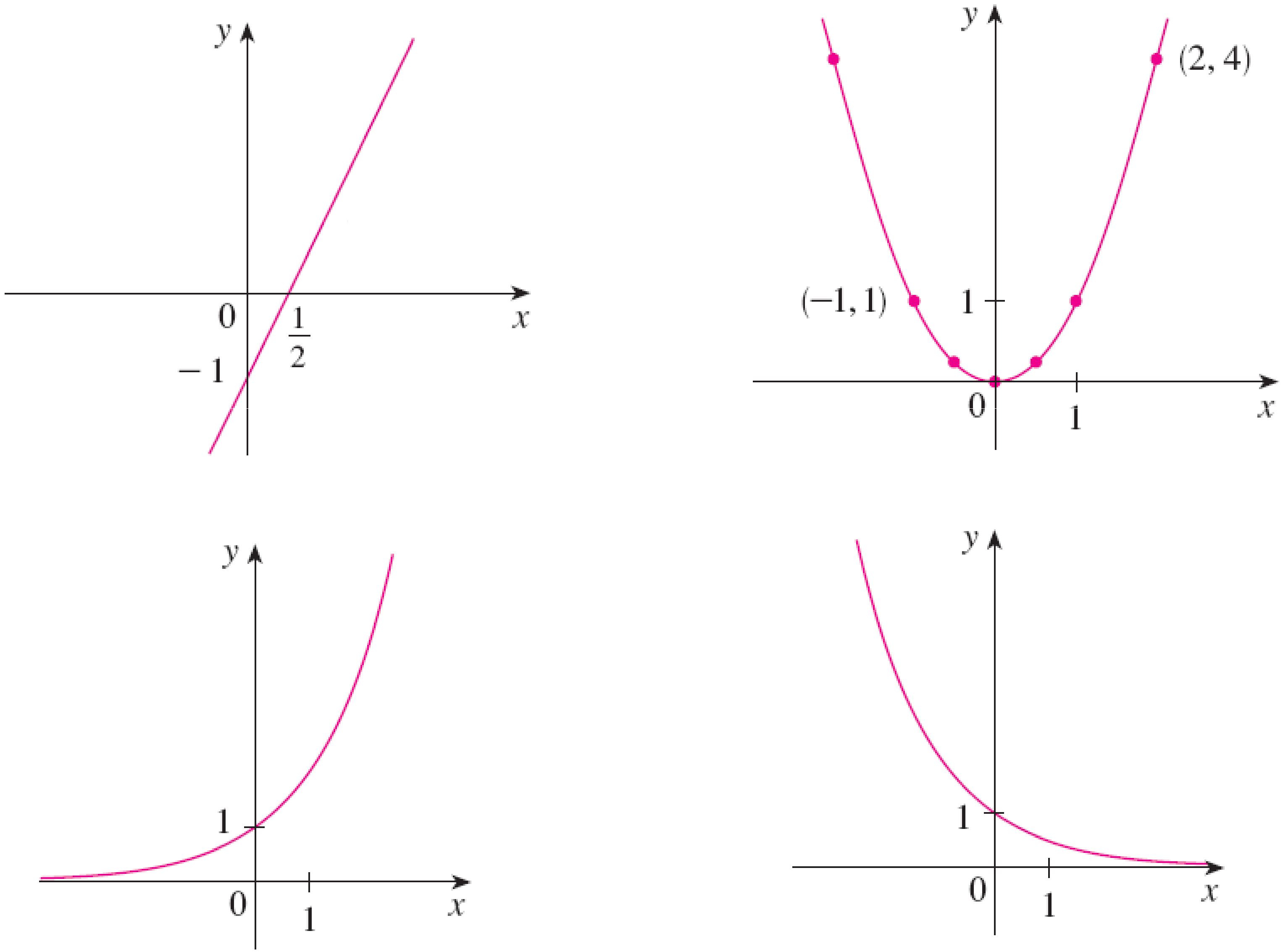
CURVES

Curves

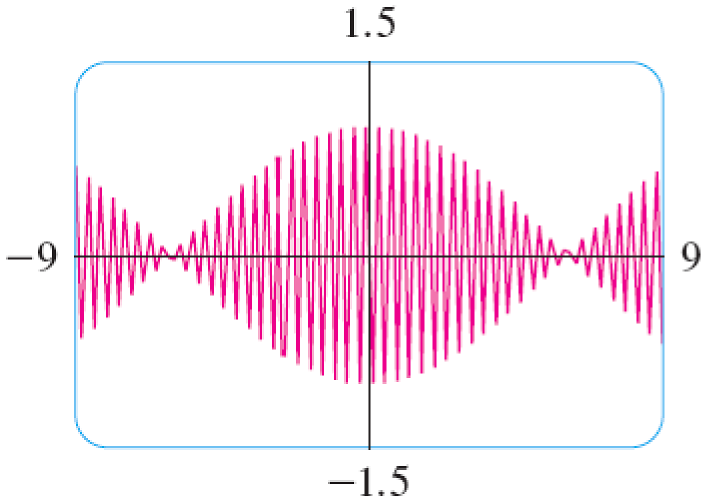
- A curve is, generally speaking, an object similar to a line but which is not required to be straight.

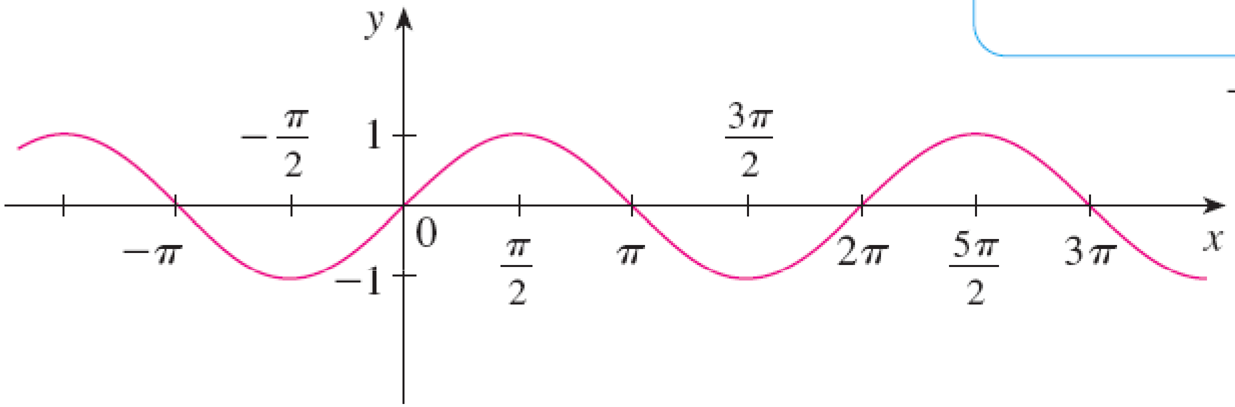


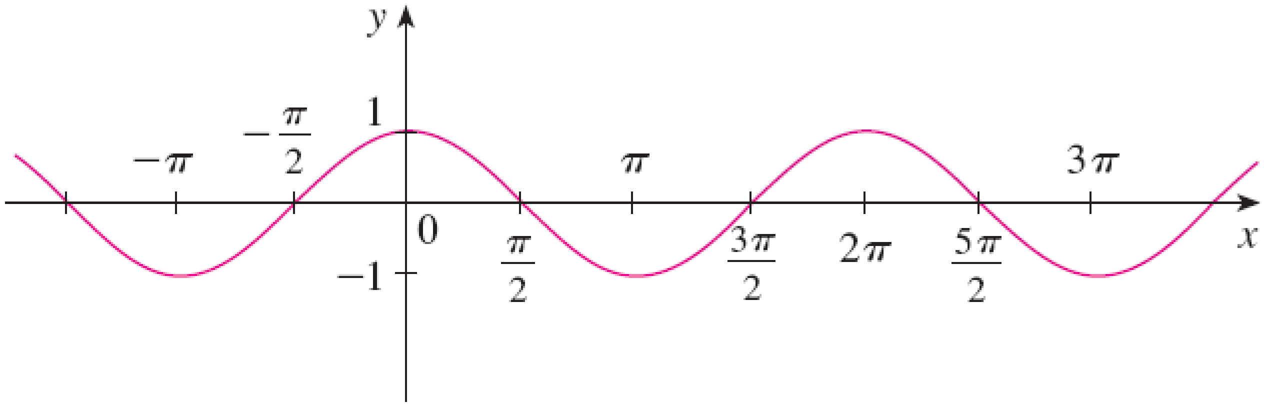
Curves



Curves

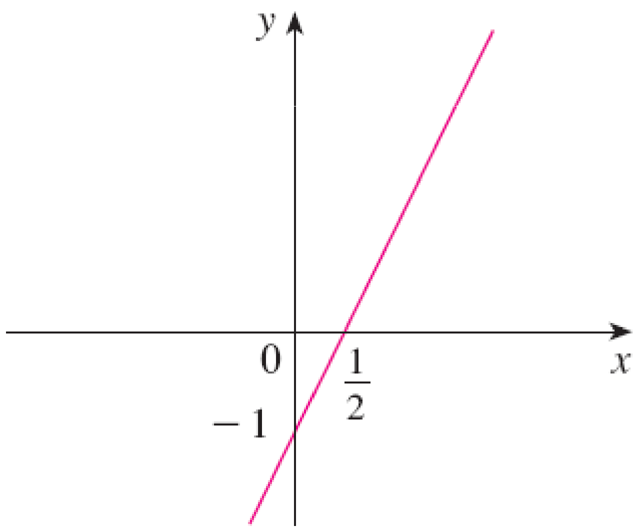

1.5
−9
9
−1.5


y
x
1
−1
0
−π
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$
3π

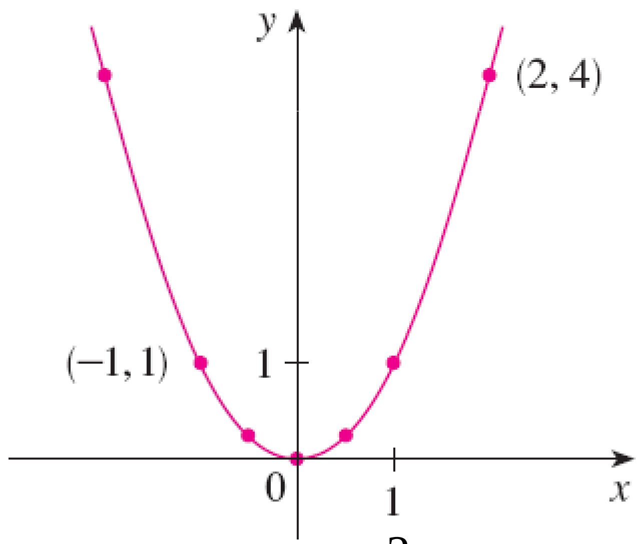

y
x
1
−1
0
−π
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$
3π

How can we describe a curve?

- Giving *y* as a function of *x*
  - $y = f(x)$



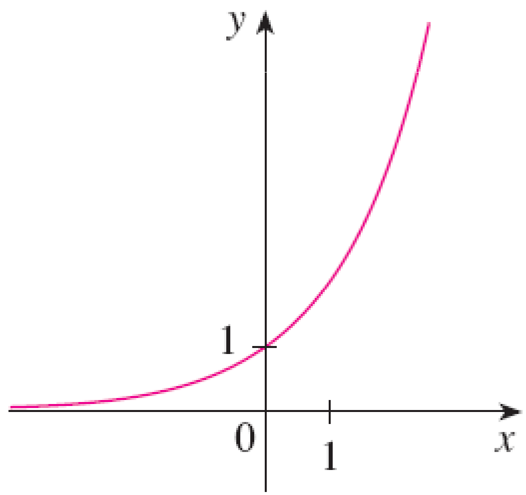
$$y = 2x - 1$$

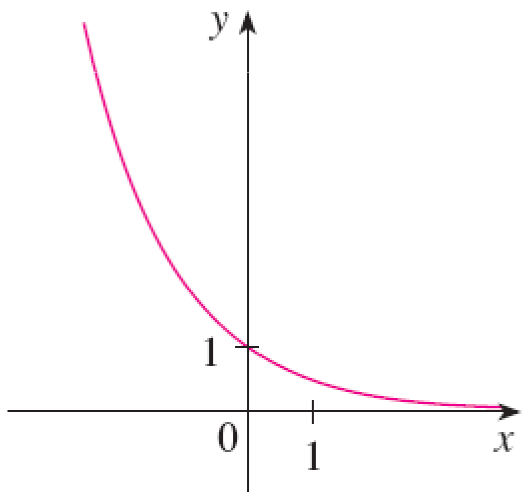


$$y = x^2$$

How can we describe a curve?

- Giving x as a function of y
  - $x = g(y)$



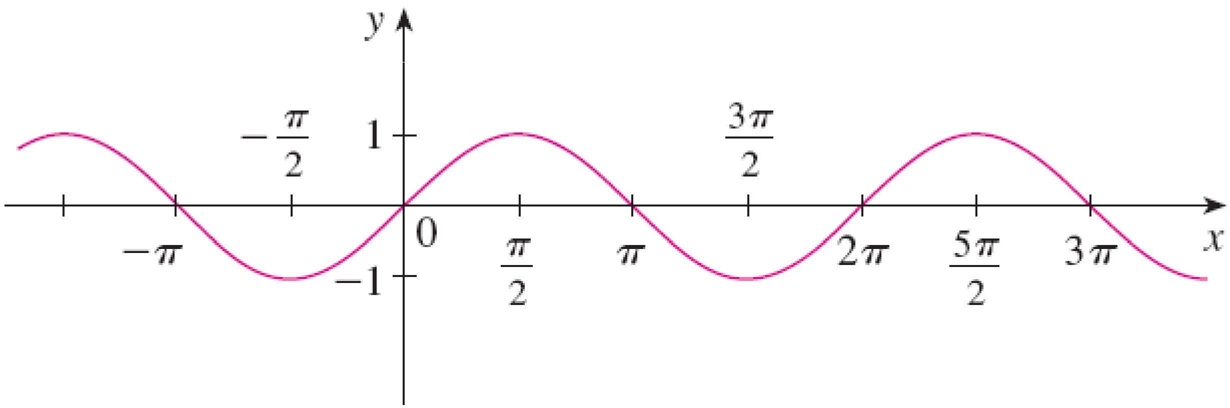


$y = 2^x \quad x = \log_2 y$

$y = 0.5^x \quad x = \log_{0.5} y$

How can we describe a curve?

- Giving a relation between *x* and *y* that defines *y* implicitly as a function of *x*
  - $f(x,y) = 0$



$$y = \sin x$$

$$f(x,y) = y - \sin x = 0$$

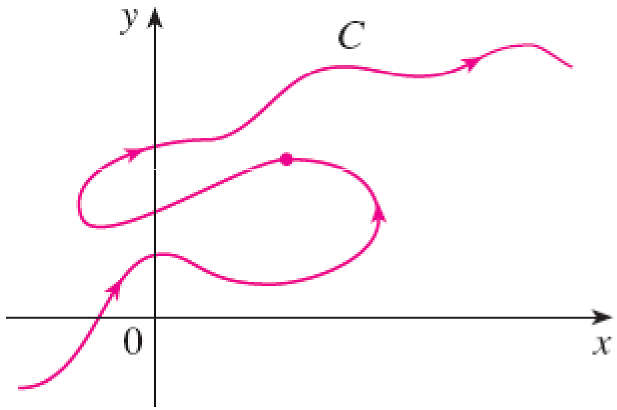
How can we describe a curve?

- Giving both x and y in terms of a third variable t called a parameter
  - $x = f(t), y = g(t)$
- Polar coordinate system
  - $r = f(\theta)$

CURVED DEFINED BY PARAMETRIC EQUATIONS

Particle movement

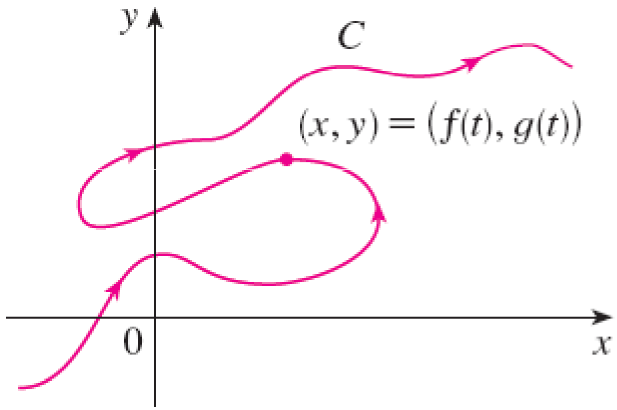
- Imagine that a particle moves along the curve C



- It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the vertical line test

Particle movement

- But x- and y-coordinates of the particle can be described as functions of time



- So we can write $x = f(t)$ and $y = g(t)$.

Particle movement

- That is, x and y are both given as functions of a third variable t.
- The variable t is called *parameter*.
- Each value of t determines a point (x,y), which we can plot in a coordinate plane.
- As t varies, the point $(x,y) = (f(t), g(t))$ varies and traces out a curve C, which we call a *parametric curve*.

Particle movement

- Note
  - In fact, the parameter does not necessarily represent time, and we can use a letter other than t for the parameter.
  - But in many applications of parametric curves, t does denote time and therefore we can interpret $(x,y) = (f(t),g(t))$ as the position of a particle at time t.

Question

- What is the advantage of parametric equations?
  - This equation in x and y describes *where* the particle has been, but it doesn't tell us *when* the particle was at a particular point.
  - The parametric equations have an advantage—they tell us *when* the particle was at a point. They also indicate the *direction* of the motion.

Example 1

- Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \quad y = t + 1$$

Example 2

- What curve is represented by the following parametric equation?

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$

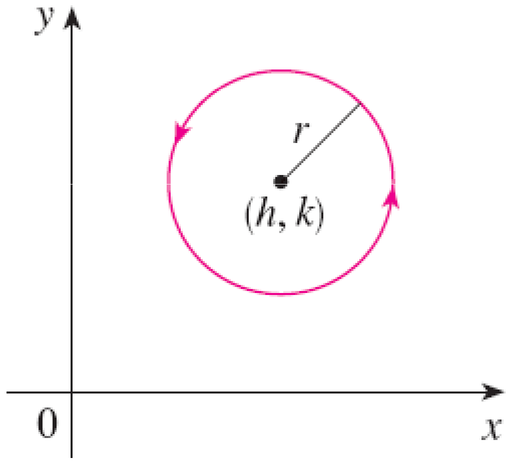
Example 3

- What curve is represented by the given parametric equation?

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq 2\pi$$

Example 4

- Find parametric equations for the circle with center *(h,k)* and radius *r*.



Example 5

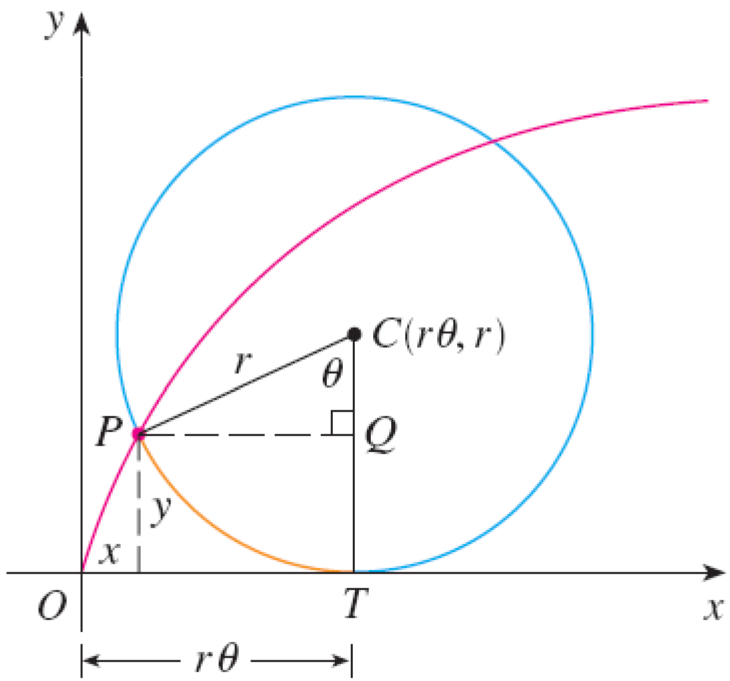
- Sketch the curve with parametric equations $x = \sin t, y = \sin^2 t$.

Example 6

- Use a graphing device to graph the curve $x = t^4 - 3t^2$ and $y = t$.

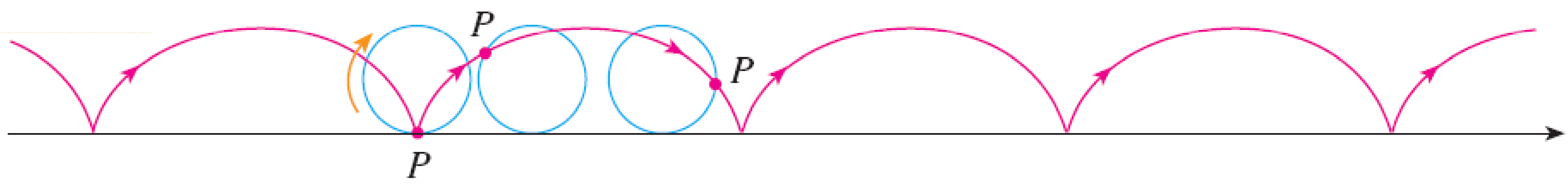
Example 7

- The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid.



Example 7 (cont')

- If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.



Example 8

- Investigate the family of curves with parametric equations

  $$x = a + \cos t \qquad y = a\tan t + \sin t$$

  What do these curves have in common? How does the shape change as a increase?

Example 8

CALCULUS WITH PARAMETRIC CURVES

Slope and tangent

- We have seen that some curves defined by parametric equations $x = f(t)$ and $y = g(t)$ can also be expressed in the form $y = F(x)$.
- If we substitute $x = f(t)$ and $y = g(t)$ in the equation $y = F(x)$, we get

$$g(t) = F(f(t))$$

Slope and tangent

- If g, F, and f are differentiable, the Chain rule gives

$$g'(t) = F'(f(t))f'(t) = F'(x)f'(t)$$

- If $f'(t) \neq 0$, we can solve for $F'(x)$:

$$F'(x) = \frac{g'(t)}{f'(t)} \qquad \boxed{1}$$

- Since the slope of the tangent to the curve $y = F(x)$ at $(x, F(x))$ is $F'(x)$, Equation 1 enables us to find tangents to parametric curves without having to eliminate the parameter.

Slope and tangent

- Using Leibniz notation, we can rewrite Equation 1 in an easily remembered form:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0 \qquad \boxed{2}$$

  - Horizontal tangent when $dy/dt = 0$ (provided that $dx/dt \neq 0$)
  - Vertical tangent when $dx/dt = 0$ (provided that $dy/dt \neq 0$)

Slope and tangent

- Also, it is useful to consider the second order derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

- Note that

$$\frac{d^2y}{dx^2} \neq \frac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}}$$

Example I

- A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.
  1. Show that C has two tangents at the point $(3,0)$ and find their equations.
  2. Find the points on C where the tangent is horizontal or vertical.
  3. Determine where the curve is concave upward or downward.

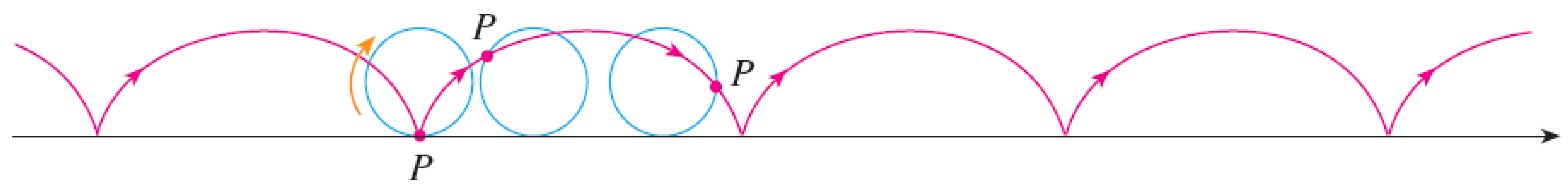
Example I

- A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

4. Sketch the curve.

Example 2

- Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point $\theta = \pi/3$.
- At what points is the tangent horizontal? When is it vertical?



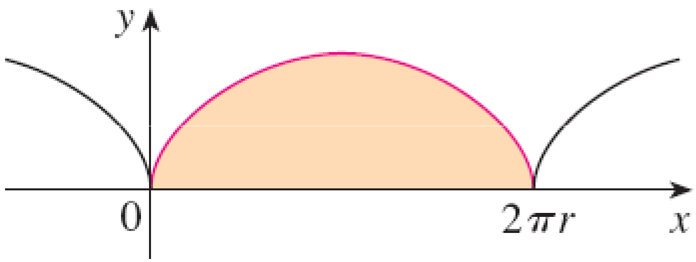
Area

- We knows that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$, where $F(x) \geq 0$.
- If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the substitution rule for definite integrals as follows:

$$A = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) dt$$

Example 3

- Find the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$



Arc length

- We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leq x \leq b$. If F' is continous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \boxed{3}$$

- Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $dx/dt = f'(t) > 0$.
- This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$.

Arc length

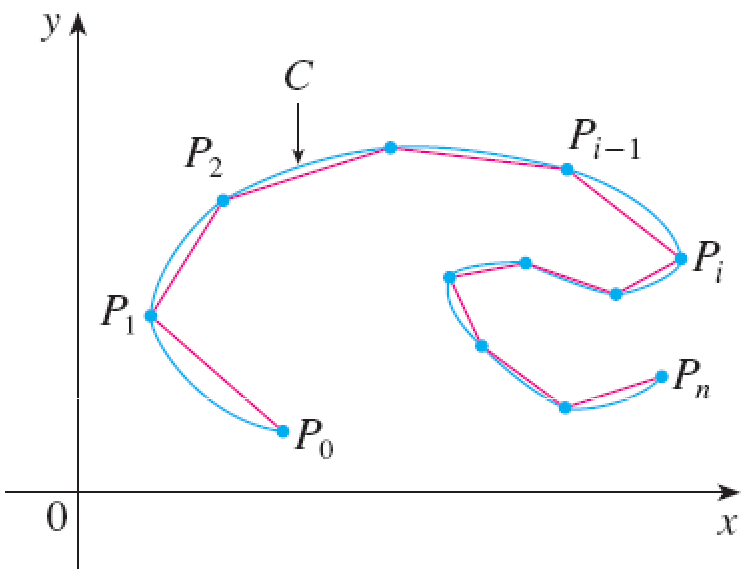
- Putting Equation 2 into Equation 3 and using the substitution rule, we have

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

- Since $dx/dt > 0$, we have

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \boxed{4}$$

Arc length



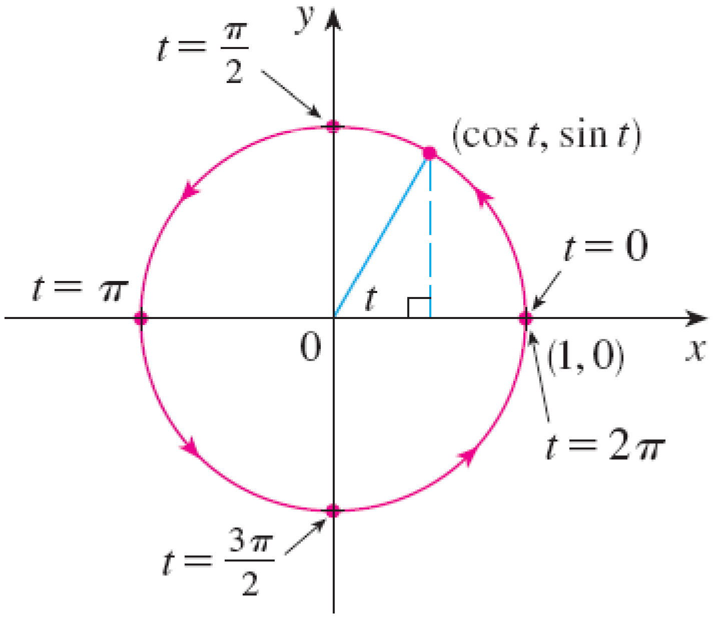
- That means we obtain the length of a curve C by polygonal approximations.

- We divide the parameter interval $[\alpha,\beta]$ into n subintervals of equal width Δt.

- If $t_0, t_1, t_2,\ldots,t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i,y_i)$ that lie one C and the polygon with vertices $P_o, P_1, \ldots, P_n$ approximates C.

Example 4

- Find the length of the unit circle described by $x = \cos t$ and $y = \sin t$, $0 \le t \le 2\pi$



Example 5

- Find the length of one arc of the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$

Surface area

- If the curve given by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, is rotated about the x-axis, where f', g' are continuous and $g(t) \geq 0$, then the area of the resulting surface is given by

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6

- Show that the surface area of a sphere of radius r is $4\pi r^2$.