Reference textbook – James Stewart, Calculus 6th ed., Thomson, 2009

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Mathematics 3

#2 Vectors and the Geometry of Space, Part I

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Content of this lecture

- Three dimensional coordinate systems
- Vectors
- The dot product and cross product

Reference textbook – James Stewart, Calculus 6th ed., Thomson, 2009

THREE DIMENSIONAL COORDINATE SYSTEMS

Locating a point in a plane

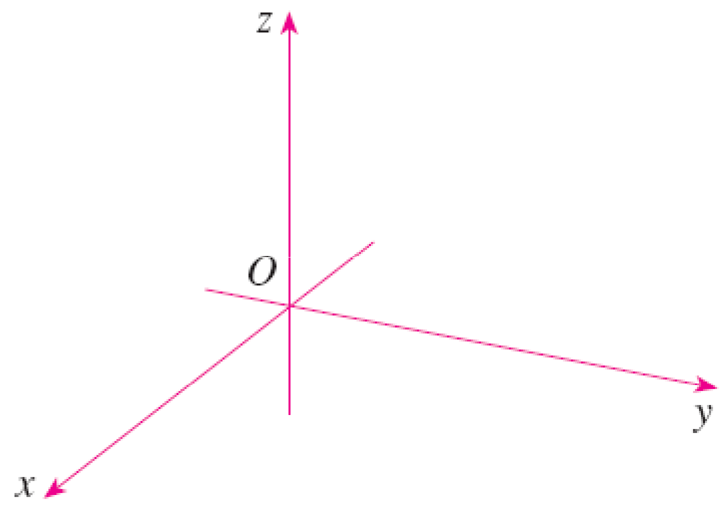
- To locate a point in a plane, two numbers are necessary.
- We know that any point in the plane can be represented as an ordered pair (a,b) of real numbers, where a is the x-coordinate and b is the y-coordinate.
- For this reason, a plane is called two-dimensional.

Locating a point in space

- To locate a point in space, three numbers are required.
- We represent any point in space by an ordered triple (a,b,c) of real numbers.
- In this case, a is the x-coordinate, b is the y-coordinate, and c is another coordinate called z-coordinate.
- That is, space is three dimensional.

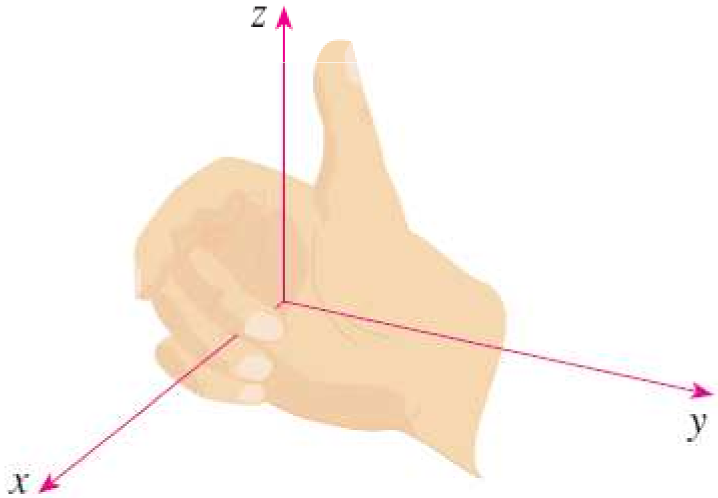
Three-dimensional coordinate systems

- To represent a point in space, we first choose a fixed point O, i.e., the origin, and three directed line through O that are perpendicular to each other, called the coordinate axes and labeled the x-axis, y-axis, and z-axis.



Three-dimensional coordinate systems

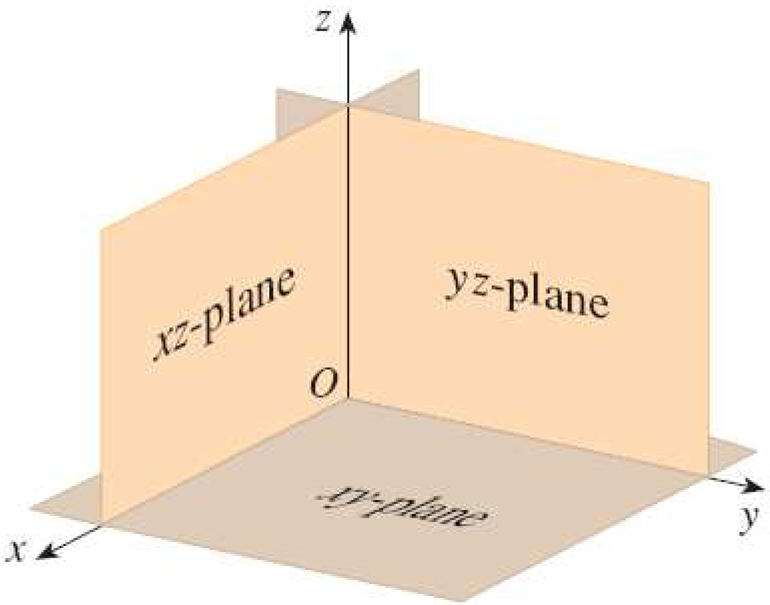
- Note that, the direction of the z-axis is determined by the right-hand rule.

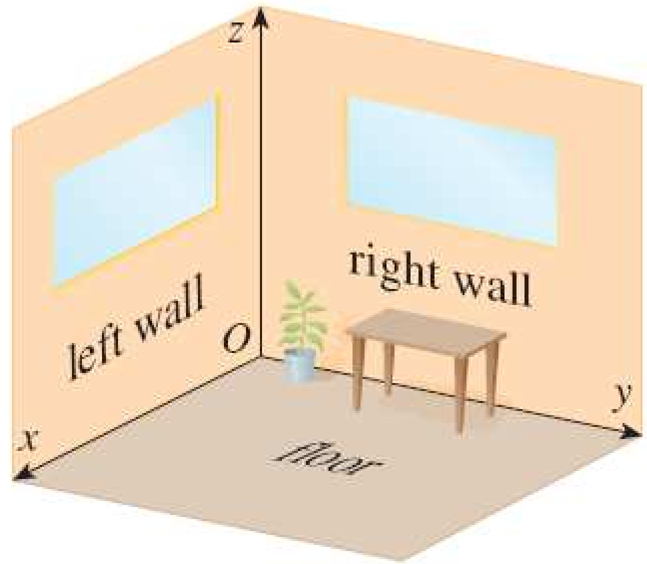


If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

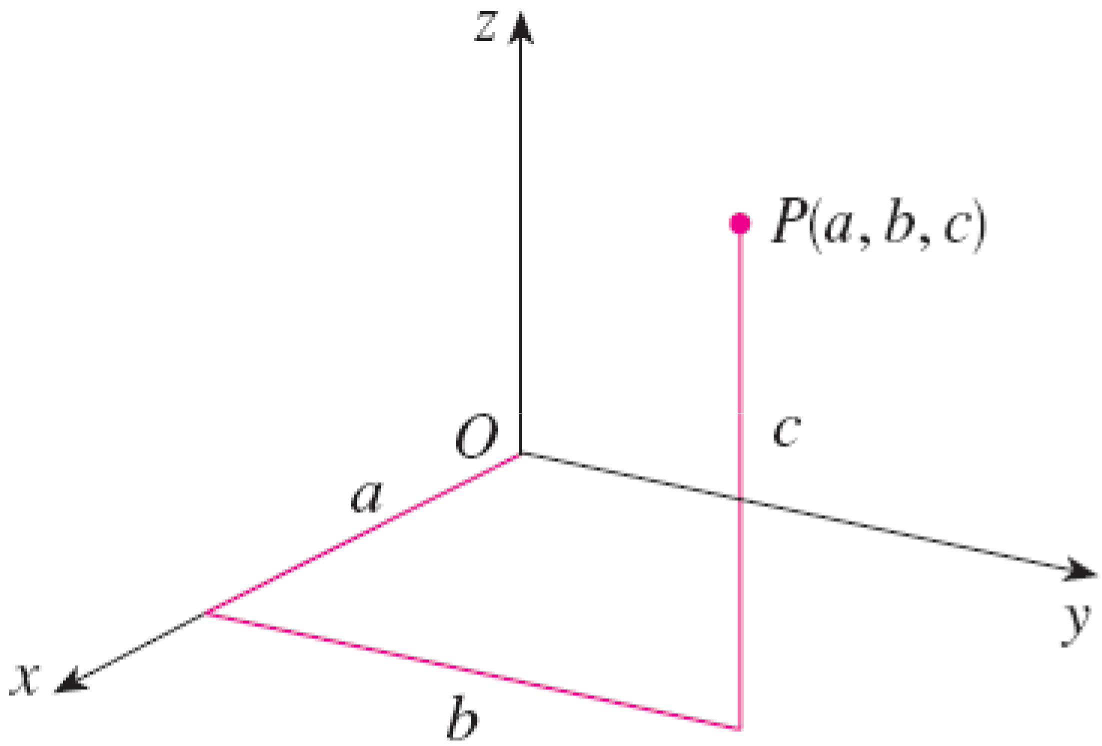
Three-dimensional coordinate systems

- The three coordinate axes determine the three coordinate planes.
- These three coordinates divide space into eight parts, called octants.





Three-dimensional coordinate systems



- The Cartesian product $\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R} = \{(x,y,z) \mid x, y, z \in \boldsymbol{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\boldsymbol{R}^3$

Three-dimensional coordinate systems

- In two-dimensional analytic geometry, the graph of equation involving x and y is a *curve* in $\boldsymbol{R}^2$.

- In three-dimensional analytic geometry, an equation in x, y, and z represents a *surface* in $\boldsymbol{R}^3$.

Example 1

- What surfaces in $\mathbf{R}^3$ are represented by the following equation?
  - $z = 3$
  - $y = 5$

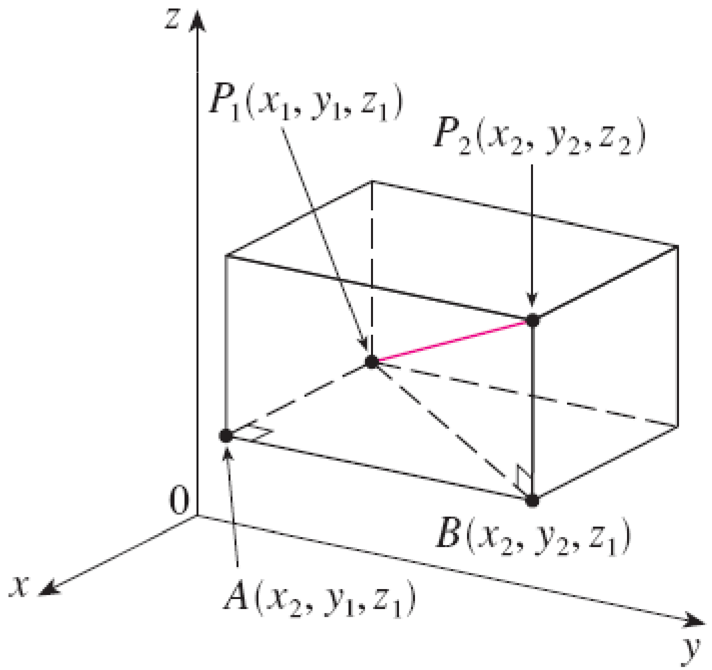
Example 2

- Describe and sketch the surface in $\boldsymbol{R}^3$ represented by the equation $y = x$.

Distance formula in three dimensions

DISTANCE FORMULA IN THREE DIMENSIONS The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

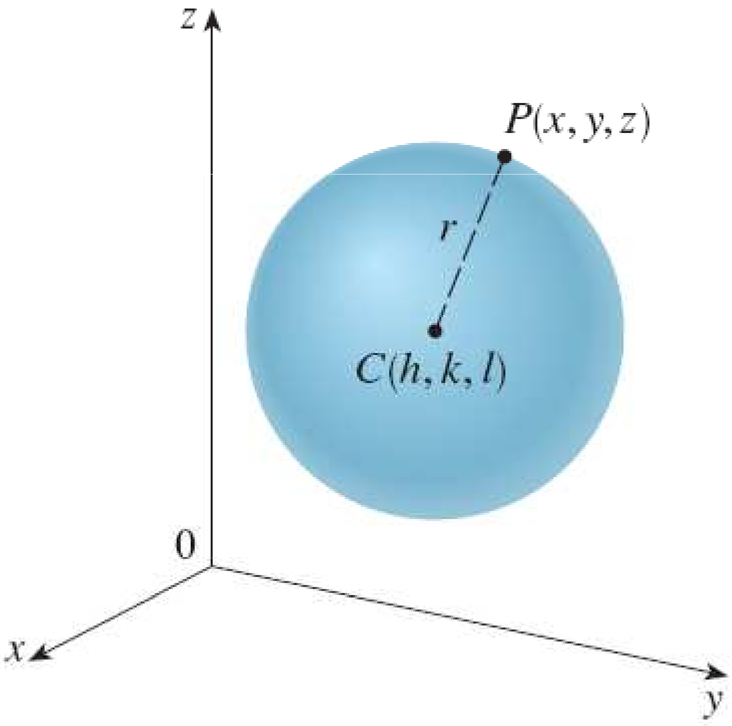


Example 3

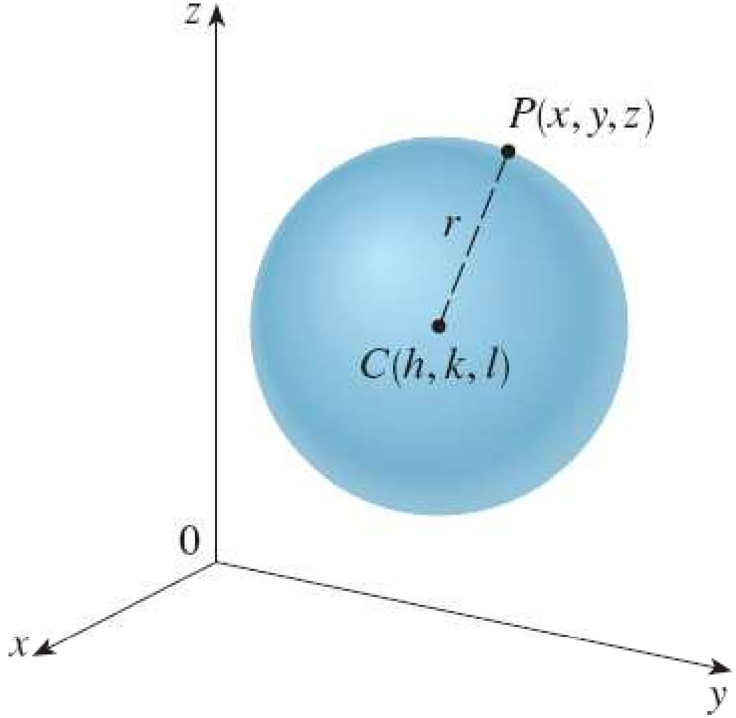
- Find the distance from the point $P(2,-1,7)$ to the point $Q(1,-3,5)$.

Example 4

- Find an equation of a sphere with radius r and center $C(h,k,l)$.



Example 4



Example 5

- Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

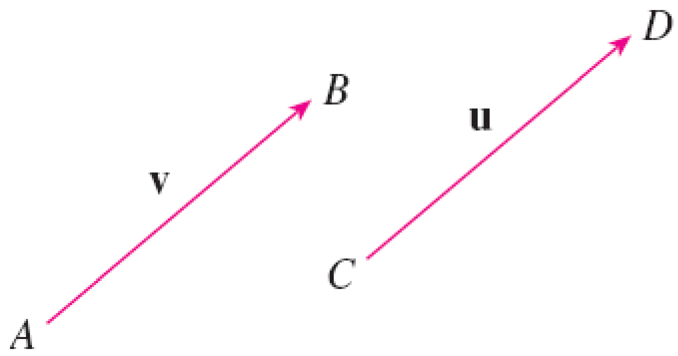
Example 6

- What region in $\boldsymbol{R}^3$ is represented by the following inequalities?

$$1 \le x^2 + y^2 + z^2 \le 4, \quad z \le 0$$

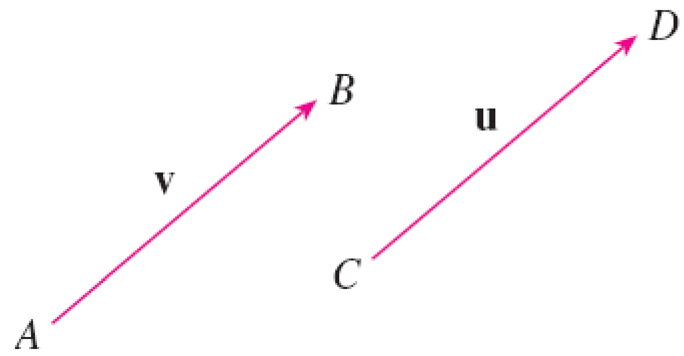
VECTORS

What is a vector?



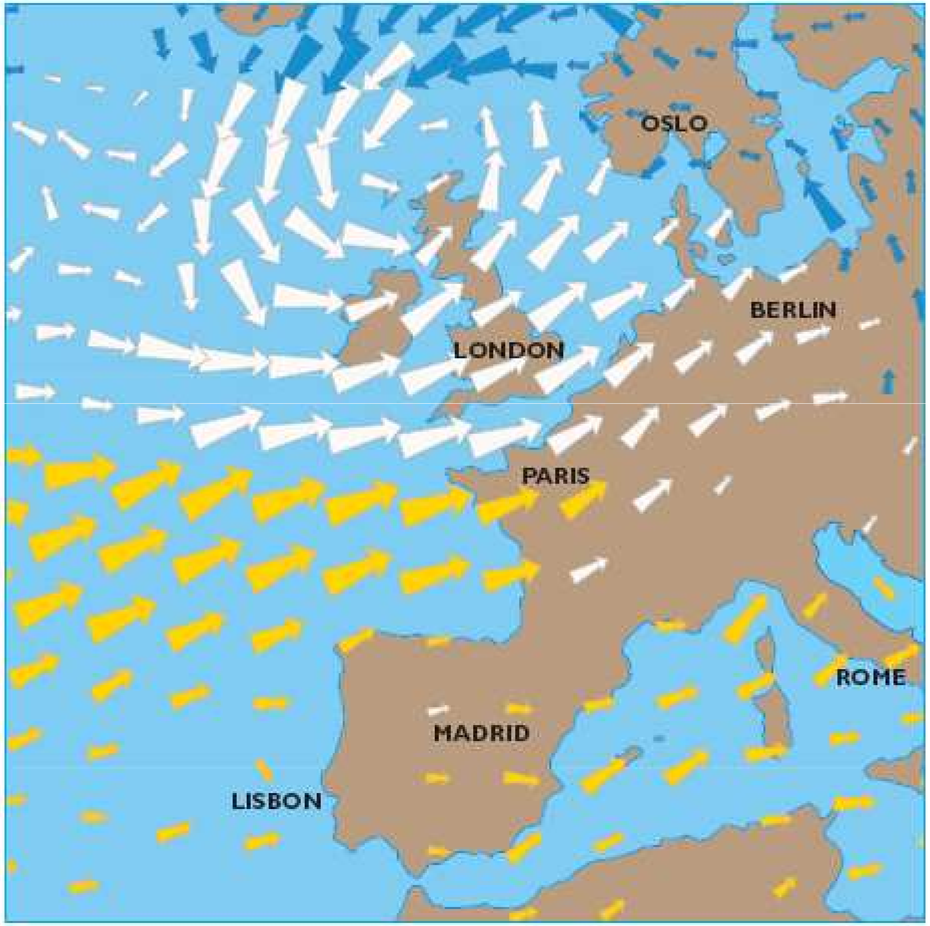
- The term vector indicates a quantity that has both magnitude and direciton.
- A vector is often represented by an arrow or a directed line segment.
- The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector.

What is a vector?

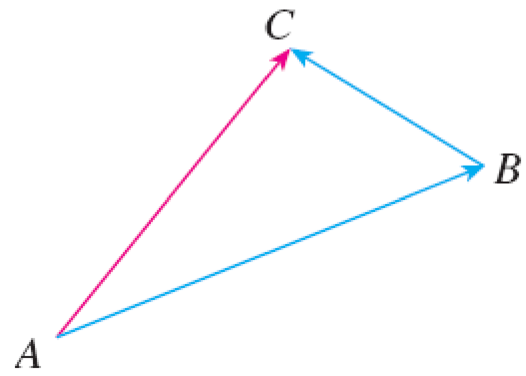


- Suppose a particle move along a line segment from A to B.
- We indicate the corresponding displacement vector v by writing $\mathbf{v} = \overrightarrow{AB}$.
- Notice that $\mathbf{u} = \overrightarrow{CD}$ has the same length and direction as $\mathbf{v}$. We say that $\mathbf{u}$ and $\mathbf{v}$ are equivalent (or equal) and we write $\mathbf{u} = \mathbf{v}$.
- The zero vector, denoted by $\mathbf{0}$, has length 0 and has no specific direction.

What is a vector?

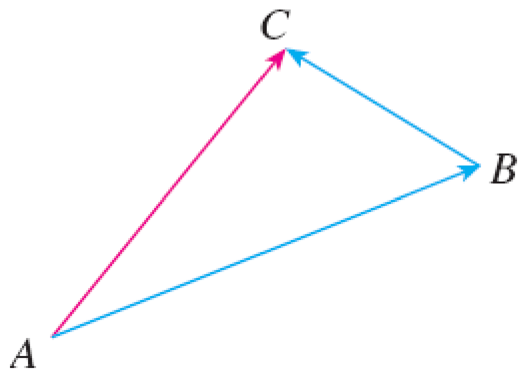


Combining vectors



- Suppose a particle moves from A to B, so its displacement vector is $\overrightarrow{AB}$.
- Then the particle changes direction and moves from B to C, with displacement vector $\overrightarrow{BC}$.
- The combined effect of these displacement is that the particle moves from A to C.

Combining vectors

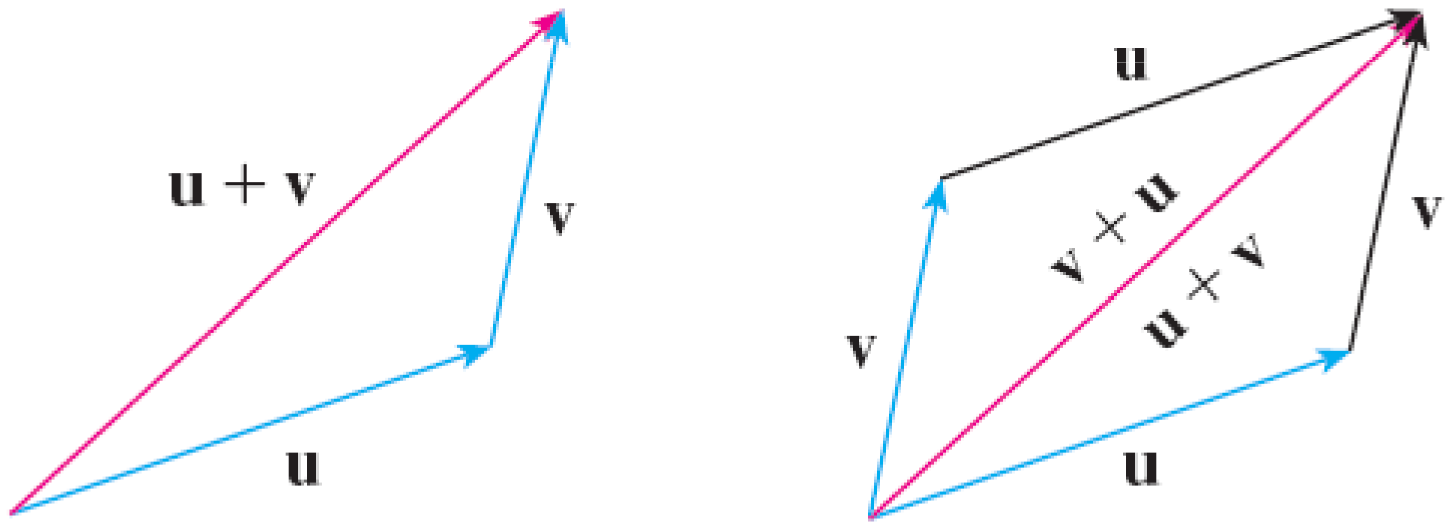


- The resulting displacement vector $\overrightarrow{AC}$ is called the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AC}$$

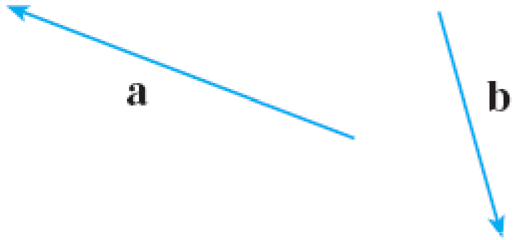
Combining vectors

DEFINITION OF VECTOR ADDITION If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u** + **v** is the vector from the initial point of **u** to the terminal point of **v**.



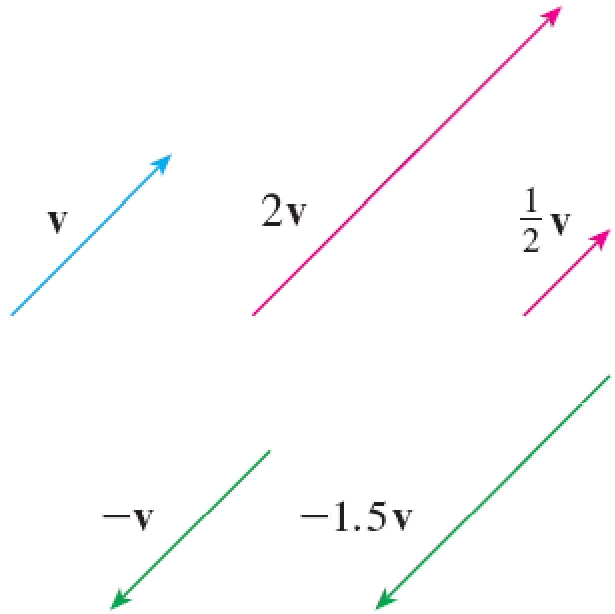
Example 1

- Draw the sum of the vectors **a** and **b**.



Scalar multiplication

- It is possible to multiply a vector with a real number c (in this context we call the real number c a scalar to distinguish it from a vector).



DEFINITION OF SCALAR MULTIPLICATION If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

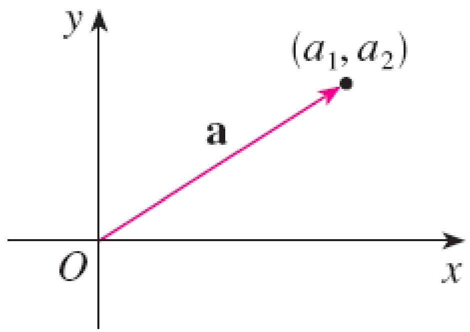
Example 2

- If **a** and **b** are the vectors shown below, draw **a** - 2**b**.

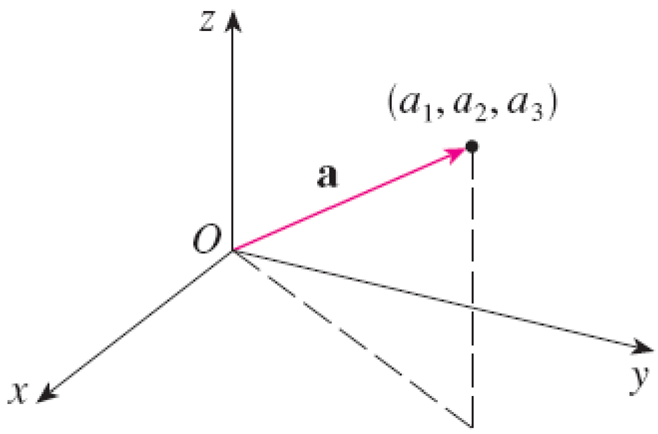
Components

- If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3). These coordinates are called the *components* of **a** and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \quad \text{or} \quad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

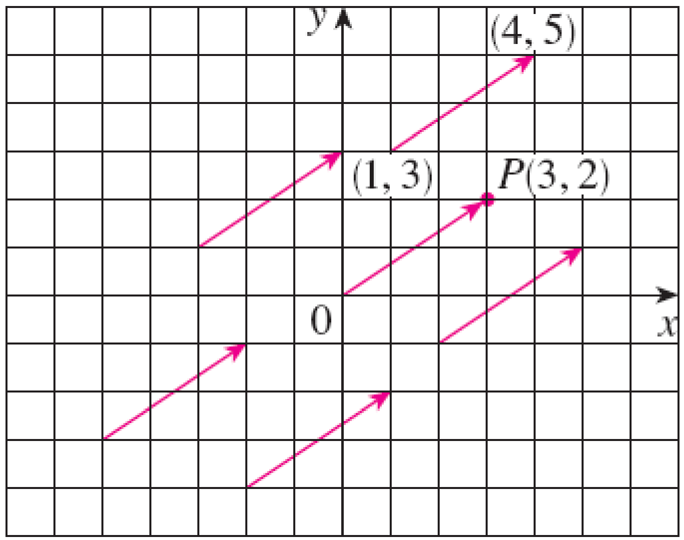


$\mathbf{a} = \langle a_1, a_2 \rangle$



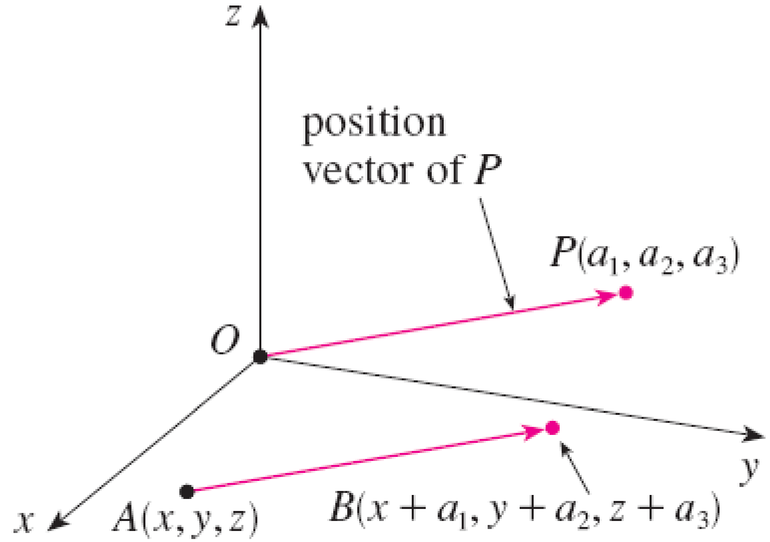
$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Components



- All of these vectors are equivalent to $\overrightarrow{OP}$.
- We can think of all these geometric vectors as *representations* of the algebraic vector $\mathbf{a} = \langle 3, 2 \rangle$.
- The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3,2)$ is called *position vector* of the point P.

Components



- In three dimensions, the vector $\mathbf{a} = \overrightarrow{OP} = <a_1, a_2, a_3>$ is the position vector of point $P(a_1, a_2, a_3)$.
- The vector $\overrightarrow{OP}$ is a representation of **a**.

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Components

- The magnitude or length of the vector $\mathbf{v}$ equals the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$.
- We can calculate the length of a vector by using the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Components

- How to add, subtract, or scalar multiply a vector algebraically?

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

n-dimensional vectors

- We denote by V_2 the set of all two-dimensional vectors and V_3 the set of all three-dimensional vectors.
- More generally, we can consider the set V_n of all n-dimensional vectors.
- An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

Properties of vectors

PROPERTIES OF VECTORS If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Standard basis vector

- Also, we can represent a vector in the term of *standard basis vectors*.
- For example, for three-dimensions,

$$\mathbf{i} = <1, 0, 0> \quad \mathbf{j} = <0, 1, 0> \quad \mathbf{k} = <0, 0, 1>$$

are the standard basis vectors.

- If $\mathbf{a} = <a_1, a_2, a_3>$, then we can write

$$\mathbf{a} = <a_1, a_2, a_3> = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

THE DOT PRODUCT AND CROSS PRODUCT

Can we multiply two vectors?

- So far we have added two vectors and multiplied a vector by a scalar.
- The question arises: Is it possible to multiply two vectors so that their product is a useful quantity?
- The answer is "Yes, it is possible".
    - ***The dot product***
    - ***The cross product***

The dot product

- The following is the definition of *the dot product*:

1 DEFINITION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- From the definition, we see that the result of the dot product is not a vector. It is a real number.
- For this reason, the dot product is sometimes called the *scalar product* (or *inner product*).

Properties of the dot product

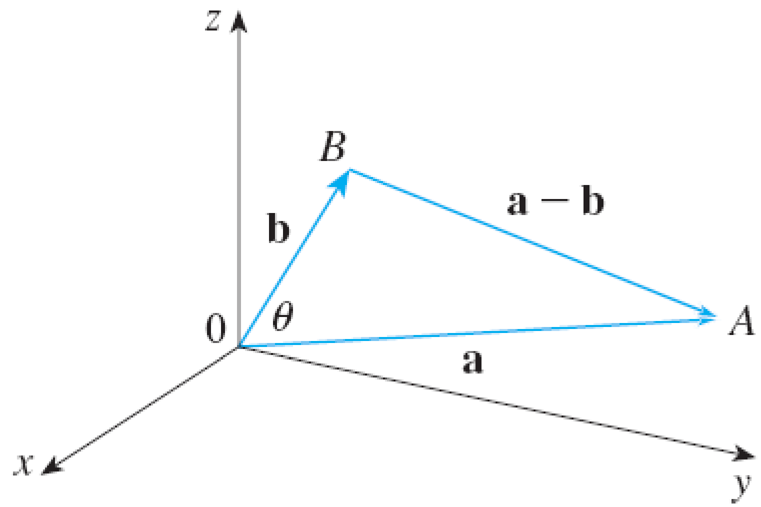
- The dot product obeys many of the laws that hold for ordinary products of real numbers.
- These are stated in the following theorem:

2 PROPERTIES OF THE DOT PRODUCT If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

The angle between two vectors

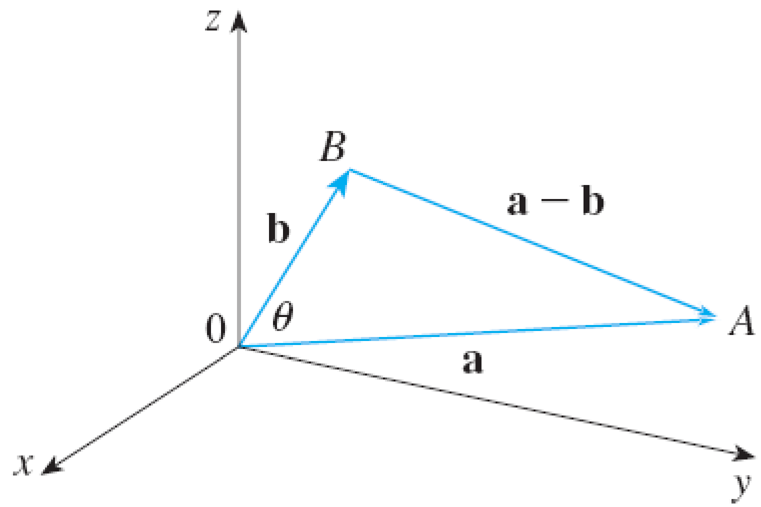
- The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the angle θ between $\mathbf{a}$ and $\mathbf{b}$.
- The following formula is used by physicist as the definition of the dot product:



3 THEOREM If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

The angle between two vectors

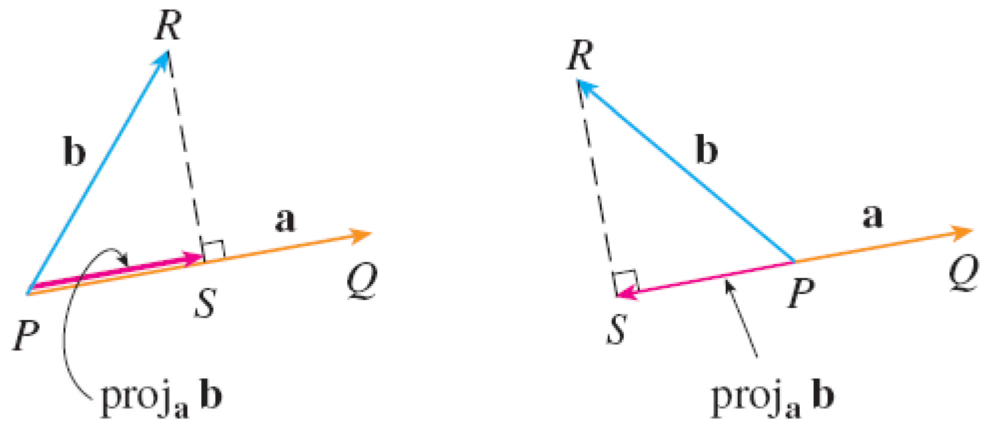


6 **COROLLARY** If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

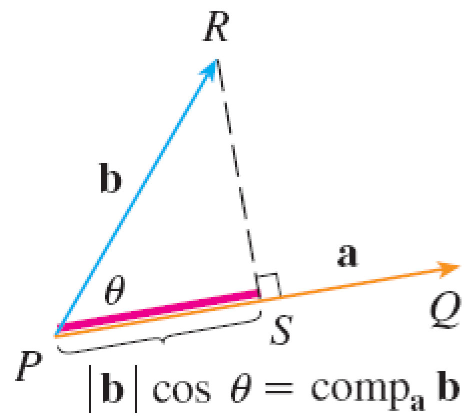
7 Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Projection



- From the figures, we have two vectors **a** and **b** with the same initial point.
- If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the vector projection of **b** onto **a** and is denoted by $proj_{\mathbf{a}}\mathbf{b}$ (we may think of it as the shadow of **b**).

Projection



- The scalar projection of **b** onto **a** (also called the component of **b** along **a**) is defined to be the signed magnitude of the vector project.

Scalar projection of **b** onto **a**: $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$

What is the cross product?

- The cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$, unlike the dot product, is a vector (for this reason, it is also called vector product).

1 DEFINITION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

- Note that $\mathbf{a} \times \mathbf{b}$ is defined only when $\mathbf{a}$ and $\mathbf{b}$ are three dimensional vectors.

What is the cross product?

- To make the definition easier to remember, we use the notation of determinants:

A determinant of order 2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A determinant of order 3

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

What is the cross product?

- Using the notation of determinant, $\mathbf{a} \times \mathbf{b}$ can be rewrite as

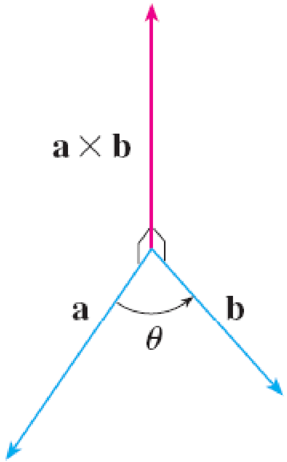
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

or

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Properties of the cross product

5 THEOREM The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.



$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3$$

$$= 0$$

Properties of the cross product

6 THEOREM If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

7 COROLLARY Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

8 THEOREM If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

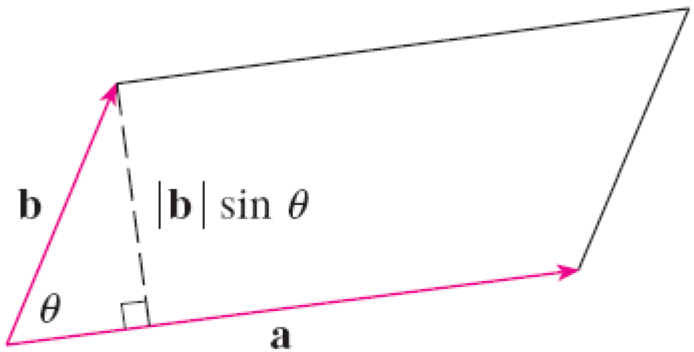
1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Geometric interpretation

- The geometric interpretation of Theorem 6 can be described by the following figure:

6 THEOREM If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

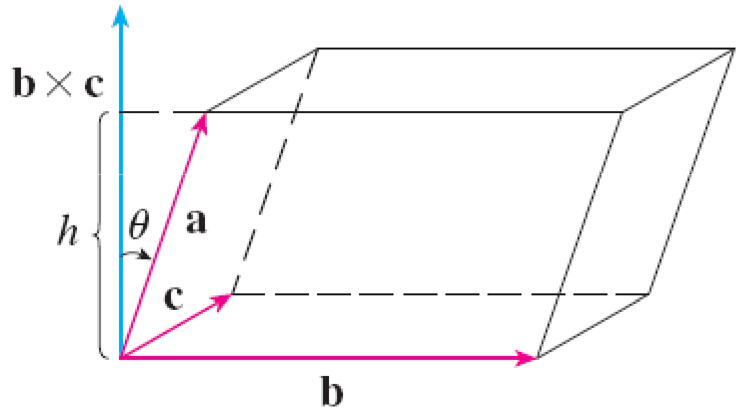


The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Triple products

- The product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is called the scalar triple product of vectors **a**, **b**, and **c**.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



11 The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$