Reference textbook – James Stewart, Calculus 6^{th} ed., Thomson, 2009

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Mathematics 3

#3 Vectors and the Geometry of Space, Part II

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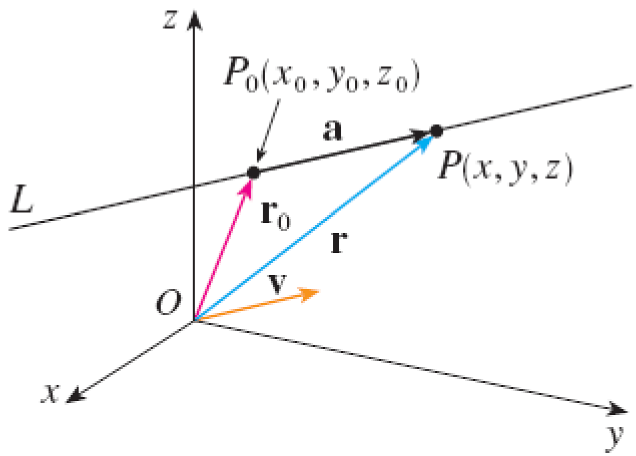
Content of this lecture

- Equations of lines
- Equations of planes
- Cylinder and quadric surfaces

Reference textbook – James Stewart, Calculus 6th ed., Thomson, 2009

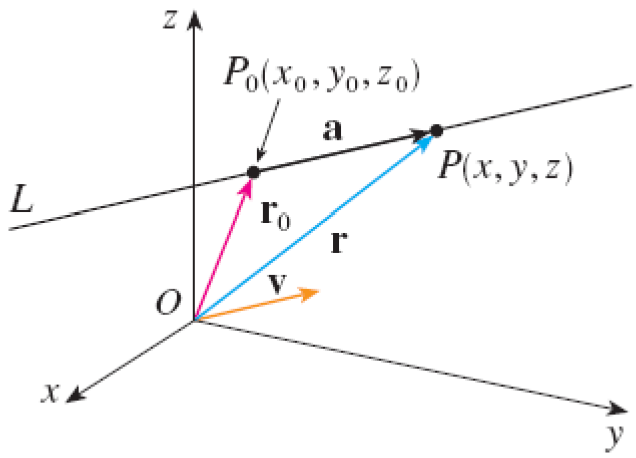
EQUATIONS OF LINES

Lines



- In three-dimensions, the direction of a line is conveniently described by a vector.
- Let $\mathbf{v}$ be a vector parallel to L.
- Let $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$ be an arbitrary point on L.
- Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P, respectively.

Lines

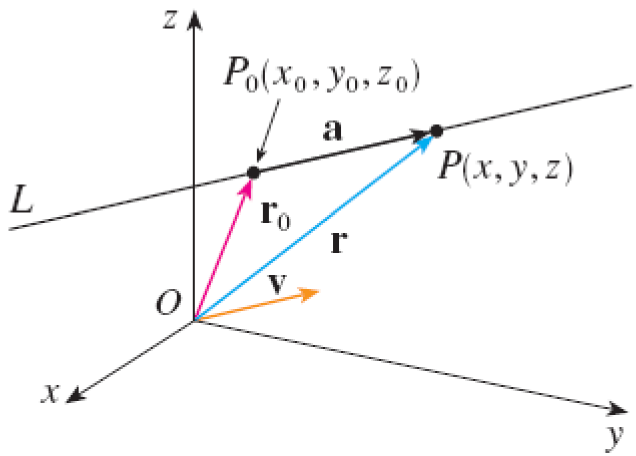


- If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_0P}$, then we get $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$
- Since $\mathbf{a}$ and $\mathbf{v}$ are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$.
- Thus,

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

which is a *vector equation* of L.

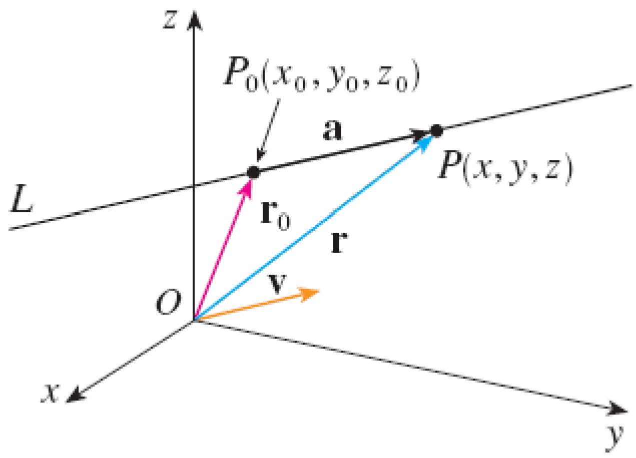
Lines



$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

- Each value of t gives the position vector $\mathbf{r}$ of a point on L.
- In other words, as t varies, the line is traced out by the tip of the vector $\mathbf{r}$.

Lines

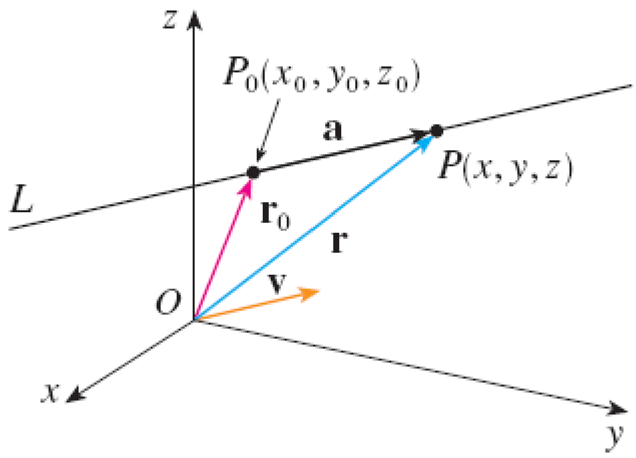


$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

- We can also write $\mathbf{v} = <a,b,c>$, $\mathbf{r} = <x, y, z>$, and $\mathbf{r}_0 = <x_0, y_0, z_0>$.
- Thus,

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Lines



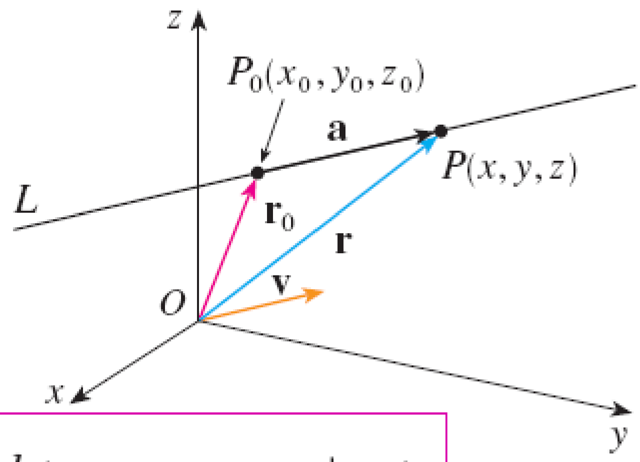
$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

- From this, we have three scalar equations:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

- These equations are called *parametric equations* of the line L through point and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$

Lines



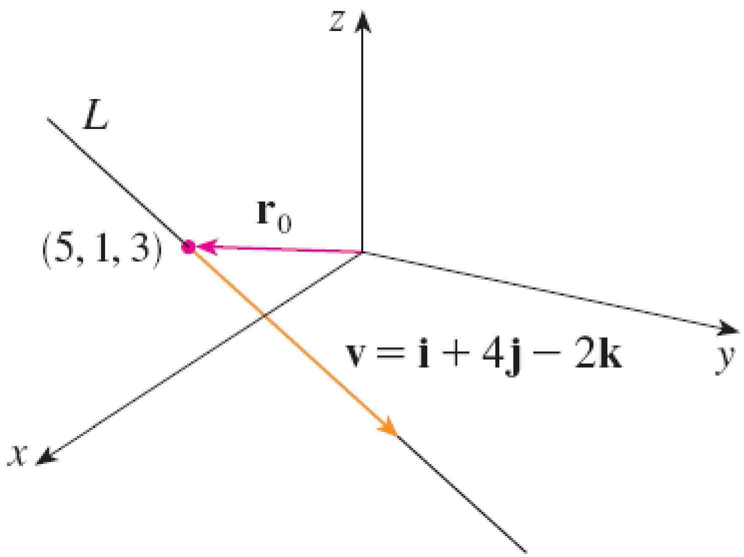
$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

- Note that:
  - The vector equation and parametric equations of a line are not unique.
  - The numbers a, b, and c are called *direction numbers* of L.
  - We can eliminate t and obtain *symmetric equations* of L:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 1

- Find a vector equation and parametric equations for the line that passes through the point $(5,1,3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
- Find the two other points on the line.



Example 2

- Find parametric equations and symmetric equations of the line that passes through the point $A(2, 4, -3)$ and $B(3, -1, 1)$.
- At what point does this line intersect the xy-plane?

Example 3

- Show that the lines L_1 and L_2 with parametric equations

  $$x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$

  $$x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

  are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

EQUATIONS OF PLANES

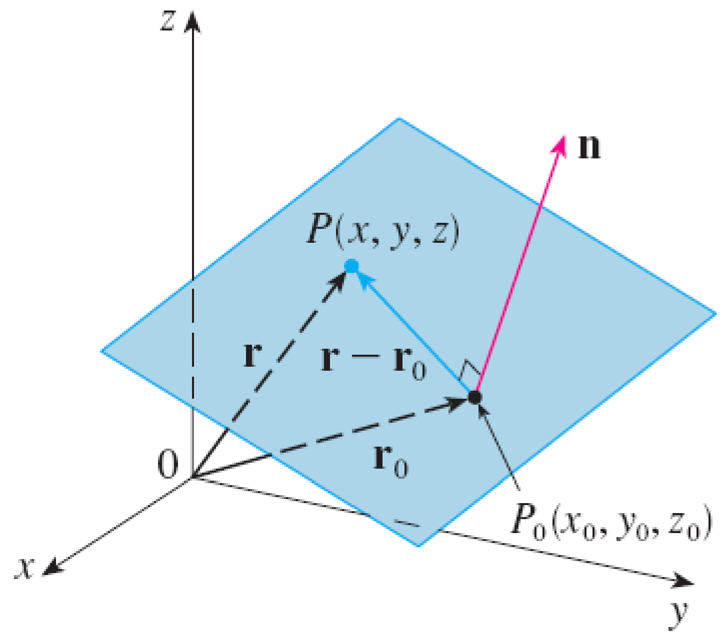
Planes

- How to represent a plane by using equation(s)?

Planes

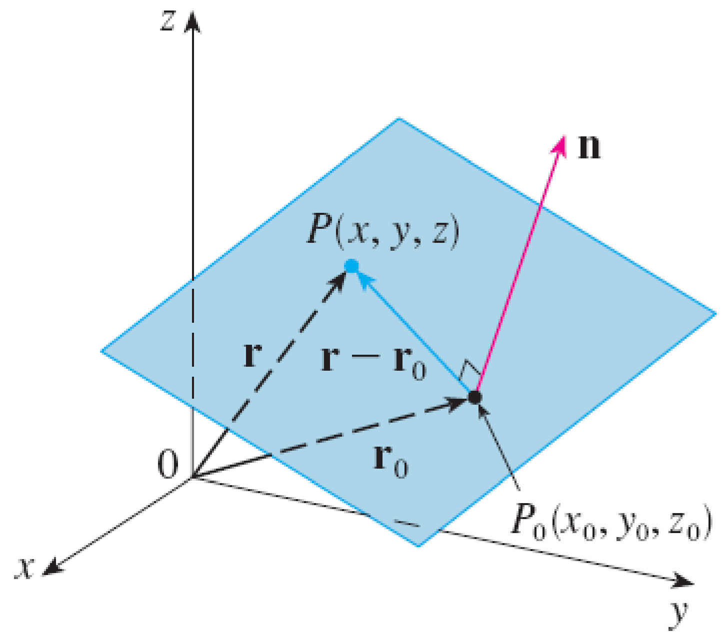
- As you learned before, a line in space is determined by a point and a direction.
- However, a plane in space is more difficult to describe.
- A single parallel vector is not enough to convey the "direction" of the plane.
- But a vector perpendicular to the plane does completely specify its direction.

Planes



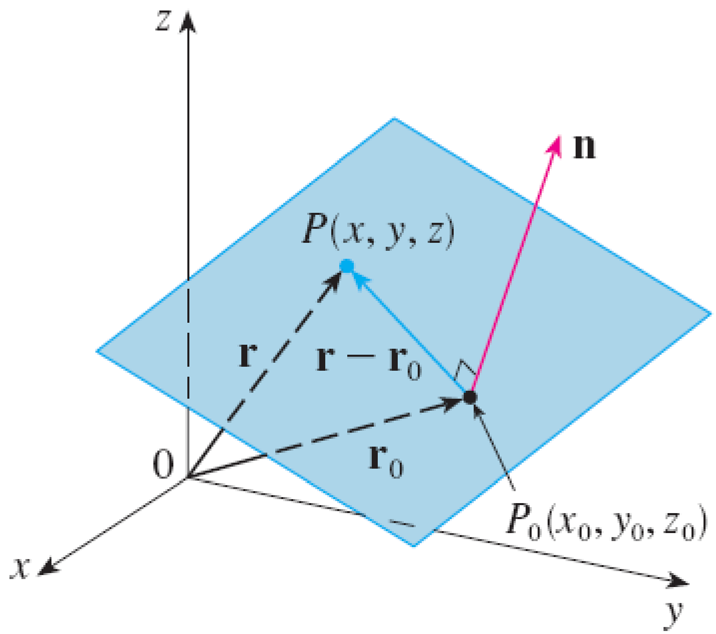
- Therefore, a plane in space can be determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector **n** that is orthogonal to the plane.
- This orthogonal vector **n** is called a *normal vector*.

Planes



- Let $P(x ,y, z)$ be an arbitrary point in the plane.
- Let $\mathbf{r}$ and $\mathbf{r}_0$ be the position vectors of P and P_0, respectively.
- Then the vector $\mathbf{r} - \mathbf{r}_0$ is also a vector in the plane.

Planes



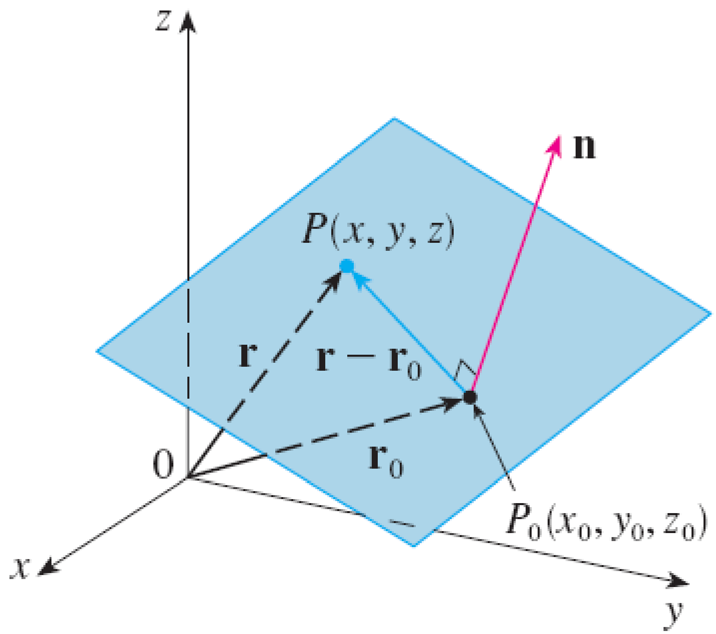
- The normal vector **n** is orthogonal to every vector in the given plane.
- Therefore, we get

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

Planes

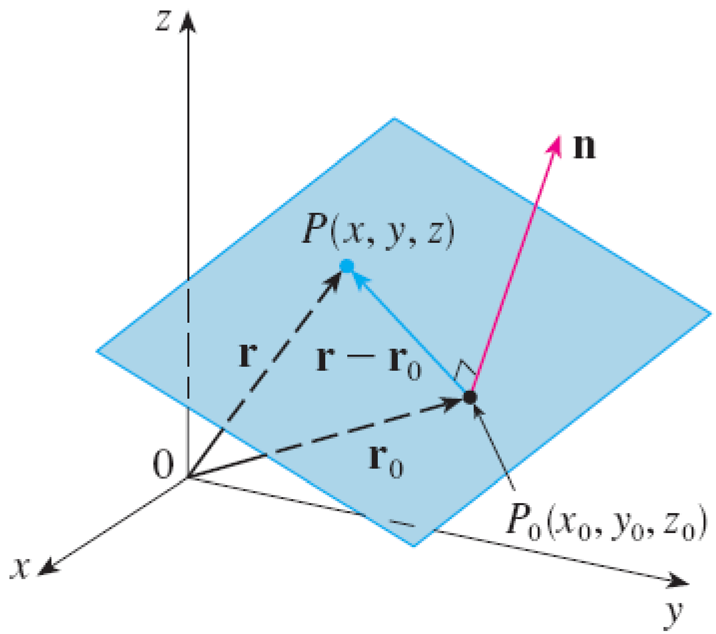


- Each of these two equations are called a *vector equation of the plane*.

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

Planes



- To obtain a scalar equation for the plane, we write $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$.
- Then the equation becomes

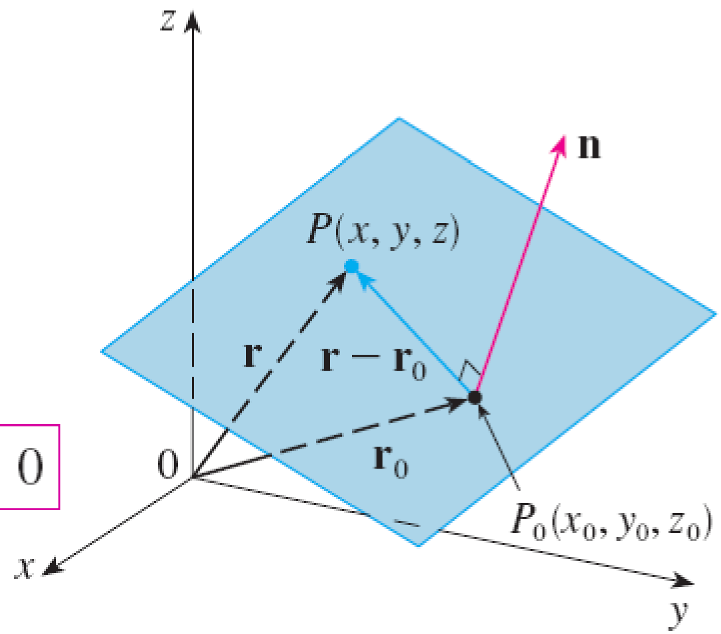
$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Planes

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



- This equation is called the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$.

Example 4

- Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $n = \langle 2, 3, 4 \rangle$.
- Find the intercepts and sketch the plane.

Linear equation in x, y, and z

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

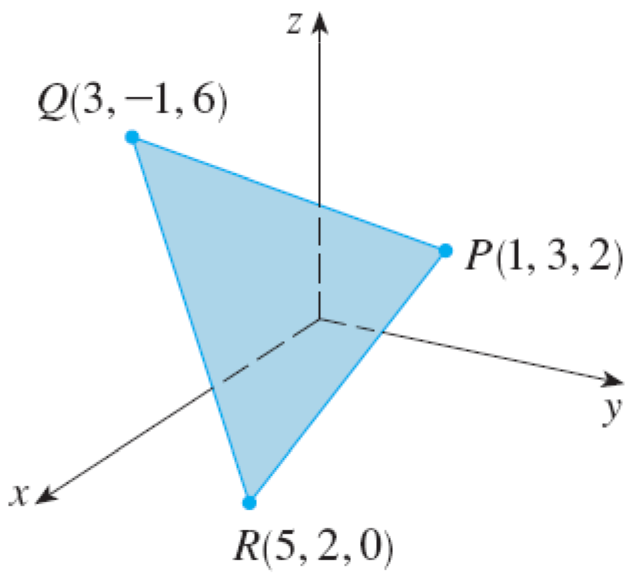
- The scalar equation of the plane can be rewritten as

$$ax + by + cz + d = 0$$

  where $d = -(ax_0 + by_0 + cz_0)$.
- This equation is called a *linear equation* in x, y, and z.
- If a, b, and c are not all 0, then the linear equation represent a plane with normal vector $<a, b, c>$.

Example 5

- Find an equation of the plane that passes through the points *P(1, 3, 2)*, *Q(3, -1, 6)*, and *R(5, 2, 0)*.



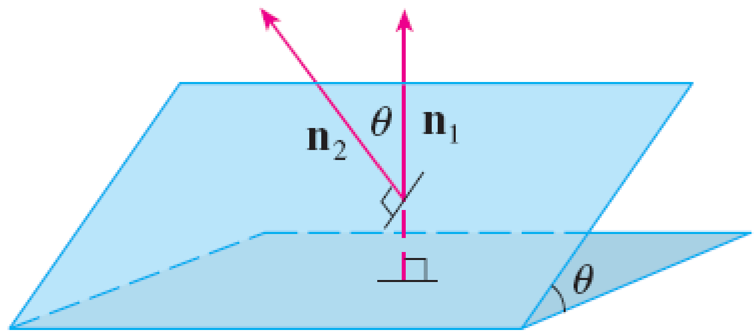
Example 6

- Find the point at which the line with parametric equations $x = 2 + 3t, y = -4t, z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

Two parallel planes

- How to check whether two given planes are parallel?

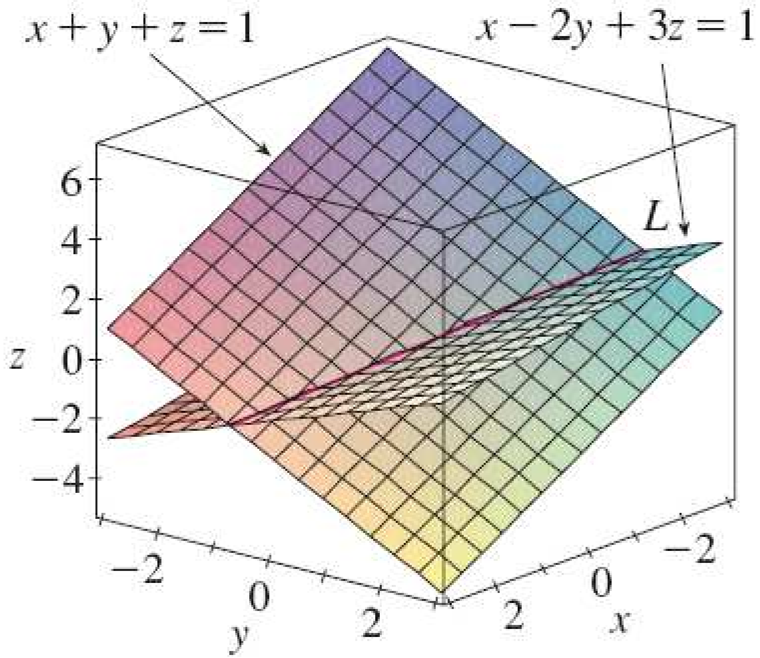
Two parallel planes



- Two planes are parallel if their normal vector are parallel.
- If two planes are not parallel, then they intersects in a straight line and the angle between the two planes is defined as the angle between their normal vectors.

Example 7

- Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
- Find the symmetric equations for the line of intersection L of those two planes.



Example 8

- Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Example 9

- Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Example 10

- In Example 3, we showed that the lines

$$L_1: x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$$

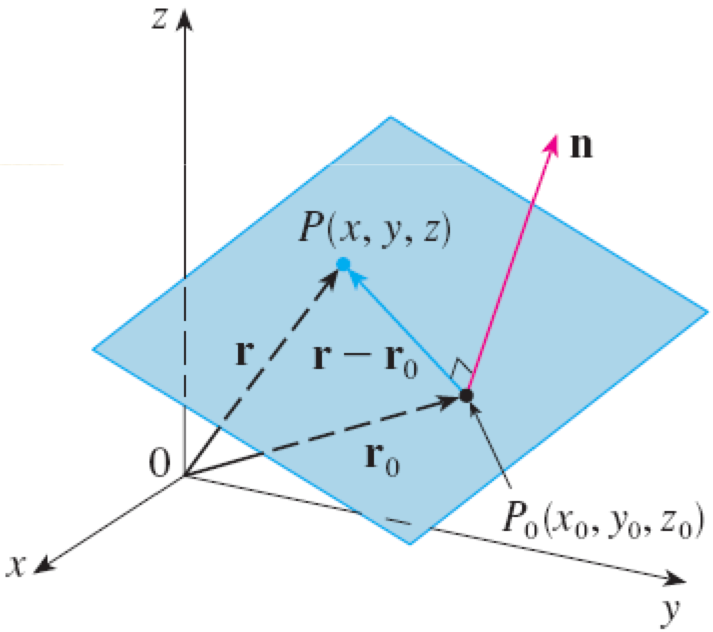
$$L_2: x = 2s \quad y = 3 + s \quad z = -3 + 4s$$

are skew. Find the distance between them.

CYLINDERS AND QUADRIC SURFACES

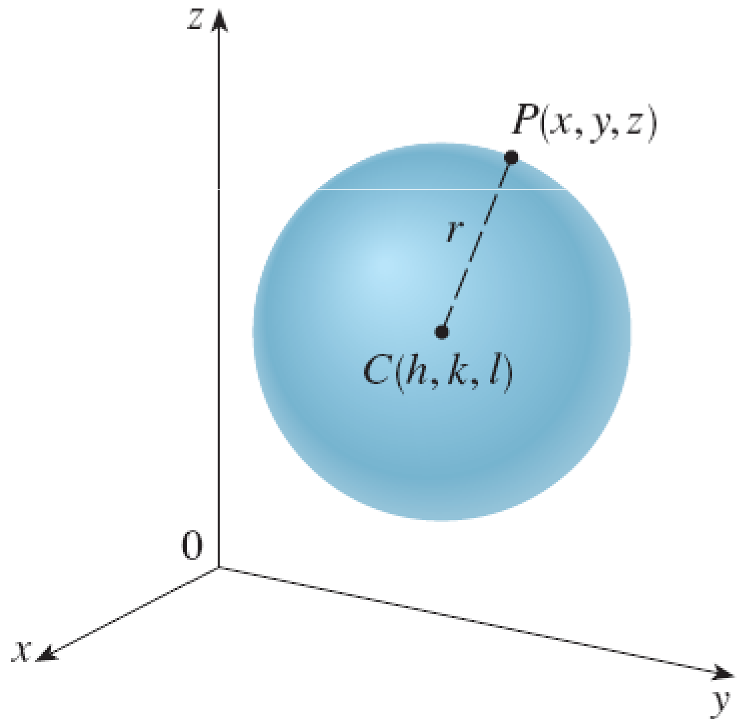
Surfaces

- So far, we have considered two special types of surfaces: planes and spheres.



$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

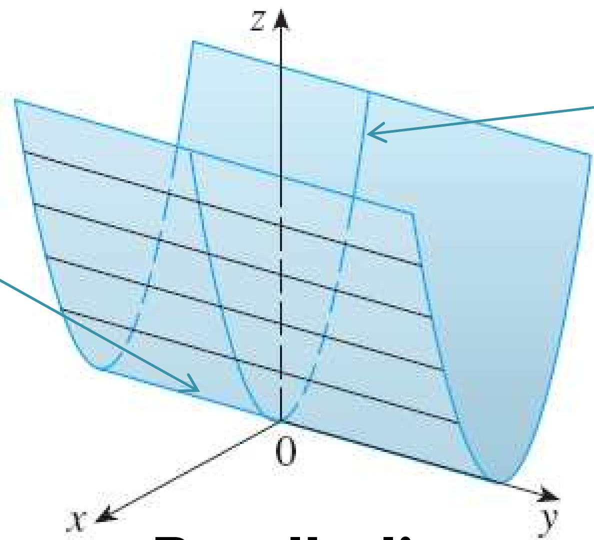


$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Cylinders

- A cylinder is a surface that consists of all lines that are parallel to a given line and pass through a given plane curve.

A given line
(parallel to y-axis)



A given plane curve
(parabola)

Paralbolic cylinder

$z = x^2$

Example 1

- Identify and sketch the surfaces

$x^2+y^2 = 1$ $y^2+z^2 = 1$

Quadric surfaces

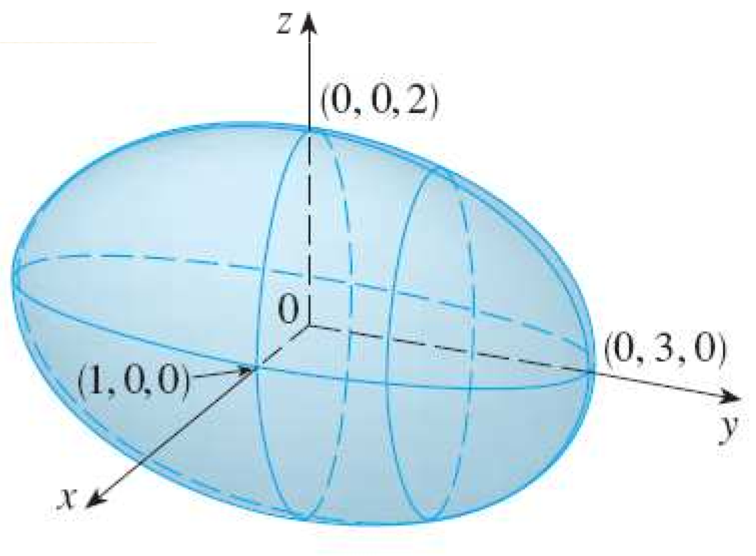
- A *quadric surface* is the graph of a second-degree equation in three variables x, y, and z.
- The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Example

- Sketch the quadric surface with equation

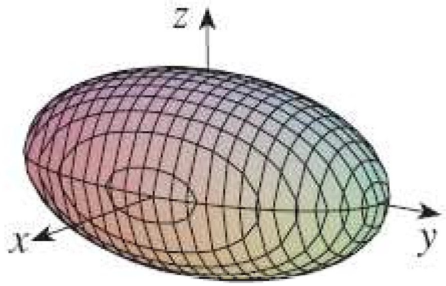
$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$



- **Trick**
  - In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes (these curves are called *traces*).

Quadric surfaces

Ellipsoid

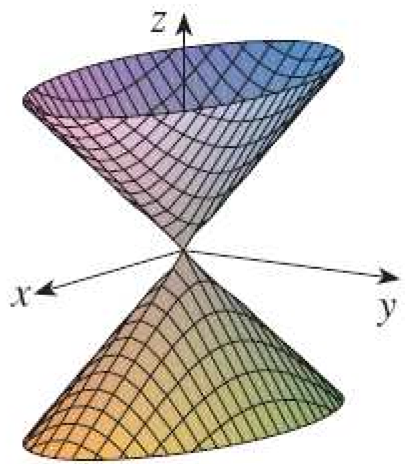


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

All traces are ellipses.

If $a = b = c$, the ellipsoid is a sphere.

Cone



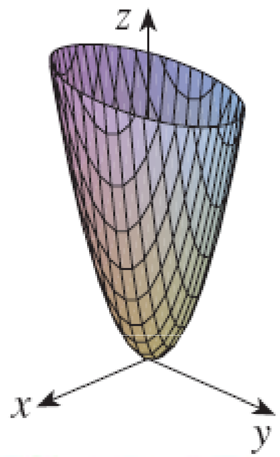
$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses.

Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

Quadric surfaces

Elliptic Paraboloid



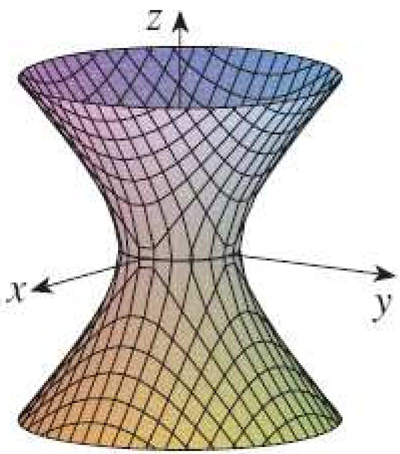
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses.

Vertical traces are parabolas.

The variable raised to the first power indicates the axis of the paraboloid.

Hyperboloid of One Sheet



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

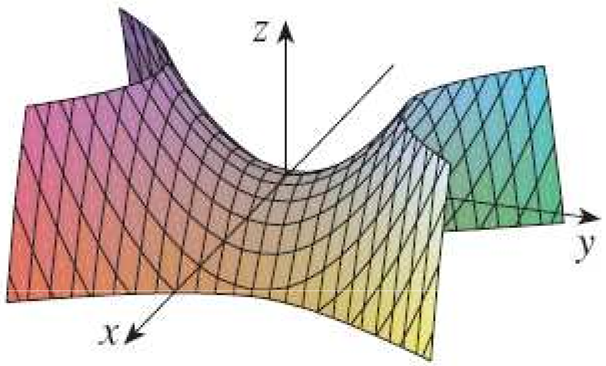
Horizontal traces are ellipses.

Vertical traces are hyperbolas.

The axis of symmetry corresponds to the variable whose coefficient is negative.

Quadric surfaces

Hyperbolic Paraboloid



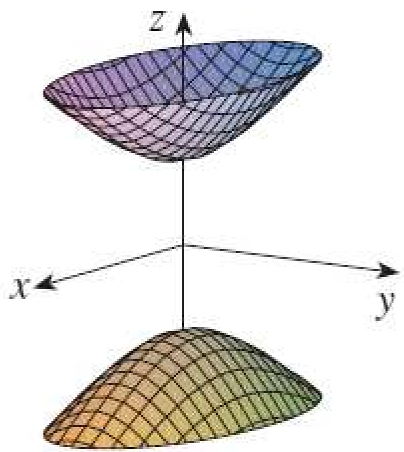
$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Horizontal traces are hyperbolas.

Vertical traces are parabolas.

The case where $c < 0$ is illustrated.

Hyperboloid of Two Sheets



$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$.

Vertical traces are hyperbolas.

The two minus signs indicate two sheets.

Applications of quadric surfaces