Reference textbook – James Stewart, Calculus 6th ed., Thomson, 2009

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Mathematics 3

#5 Vector Functions, Part II

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Content of this lecture

- Derivatives and integrals of vector functions
- Arc length and curvature
- Motion in space: velocity and acceleration

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DERIVATIVES AND INTEGRALS OF VECTOR FUNCTIONS

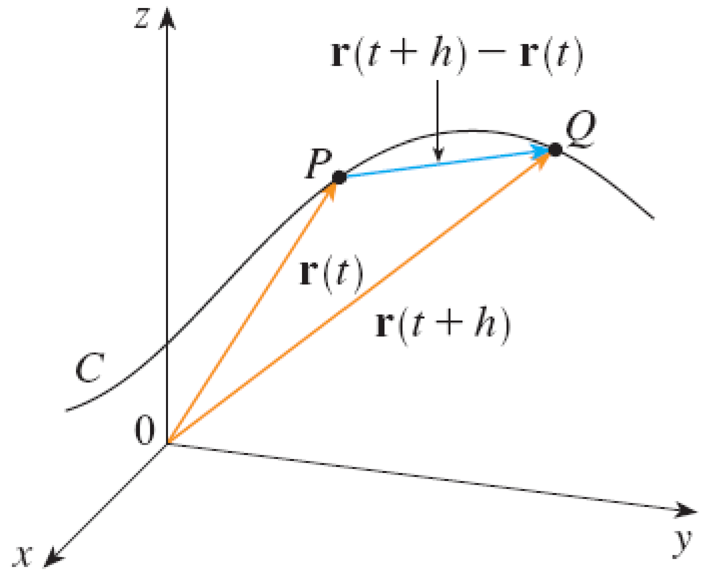
Derivatives

- The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in the same way as for real-valued functions:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

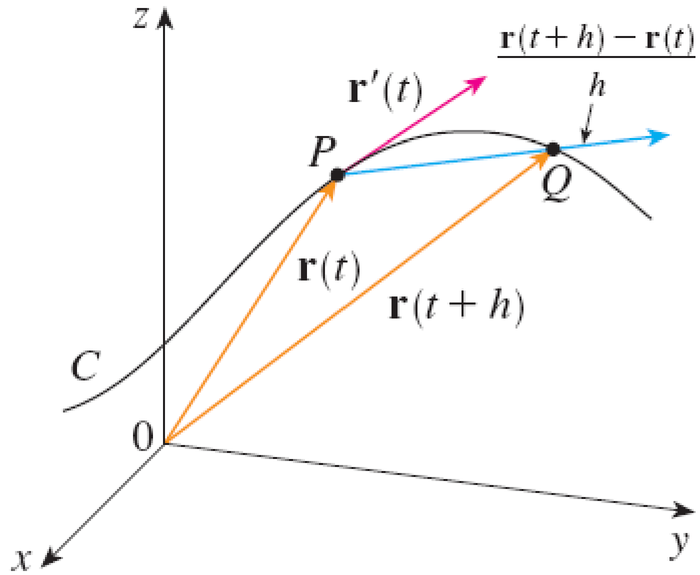
if this limit exists.

Derivatives



- The geometric significant is shown in the figure.
- The points P and Q have position vector $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$.
- The vector from P to Q is represented by $\mathbf{r}(t+h) - \mathbf{r}(t)$, regarded as a *secant vector*.

Derivatives



- If $h > 0$, the vector $(1/h)(\mathbf{r}(t+h) - \mathbf{r}(t))$ will have the same direction as $\mathbf{r}(t+h) - \mathbf{r}(t)$.
- As $h \to 0$, this vector approaches a vector that lies on the tangent line.
- So the vector $\mathbf{r}'(t)$ is called the *tangent vector*. And the *unit tangent vector* is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Derivatives

- The following theorem gives us a convenient method for computing the derivative of a vector function $\mathbf{r}$:

2 THEOREM If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

Derivatives

PROOF

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\mathbf{r}(t + \Delta t) - \mathbf{r}(t)] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\langle f(t + \Delta t), g(t + \Delta t), h(t + \Delta t)\rangle - \langle f(t), g(t), h(t)\rangle] \\
&= \lim_{\Delta t \to 0} \left\langle \frac{f(t + \Delta t) - f(t)}{\Delta t}, \frac{g(t + \Delta t) - g(t)}{\Delta t}, \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \langle f'(t), g'(t), h'(t)\rangle
\end{aligned}$$

□

Example 1

- Find the derivative of $\mathbf{r}(t) = (1+t^3)\mathbf{i} + te^{-1}\mathbf{j} + \sin(2t)\,\mathbf{k}$.
- Find the unit tangent vector at the point where $t = 0$.

Example 2

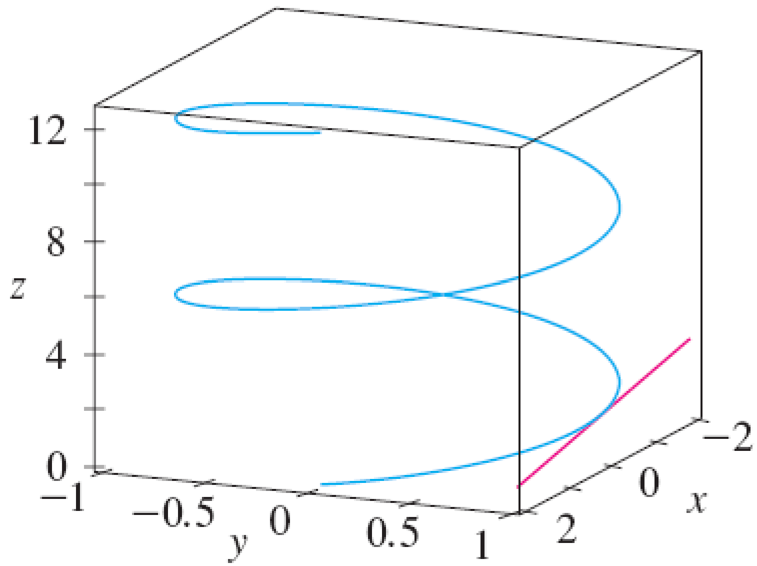
- For the curve $\mathbf{r}(t) = t^{1/2}\mathbf{i} + (2 - t)\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

Example 3

- Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$



Derivative rules

- Suppose **u** and **v** are diffentiable vector functions, c is a scalar and f is a real-valued fucntion. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Example 4

- Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Integrals

- The *definite integral* of a continuous vector function $\mathbf{r}(t)$ can be defined in the same way as for real-valued functions.

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\,\Delta t$$

$$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\,\Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\,\Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\,\Delta t \right) \mathbf{k} \right]$$

and so

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt \right) \mathbf{i} + \left(\int_a^b g(t)\,dt \right) \mathbf{j} + \left(\int_a^b h(t)\,dt \right) \mathbf{k}$$

Integrals

- We can rewritten the formular as follow:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}(t)$ is an *antiderivative* of $\mathbf{r}$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

- We use the notation $\int \mathbf{r}(t)$ for *indefinite integrals* (antiderivatives).

Example 5

- If $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}$, then find the antiderivative of $\mathbf{r}(t)$.

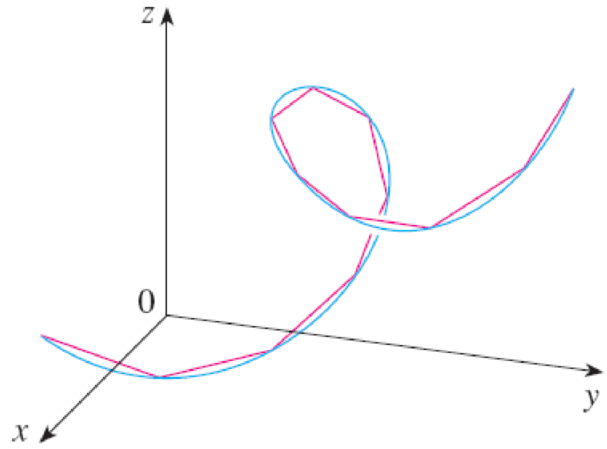
ARC LENGTH AND CURVATURE

Arc length

- In previous lecture, we have defined the length of a *plane curve* with parametric equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Arc length



- The length of a *space curve* is defined in exactly the same way:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Arc length

- Note that:
  - The arc length formulars for both *plane curves* (2D) and *space curves* (3D) can be put into the more compact form:

$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$

because, for plane curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$,

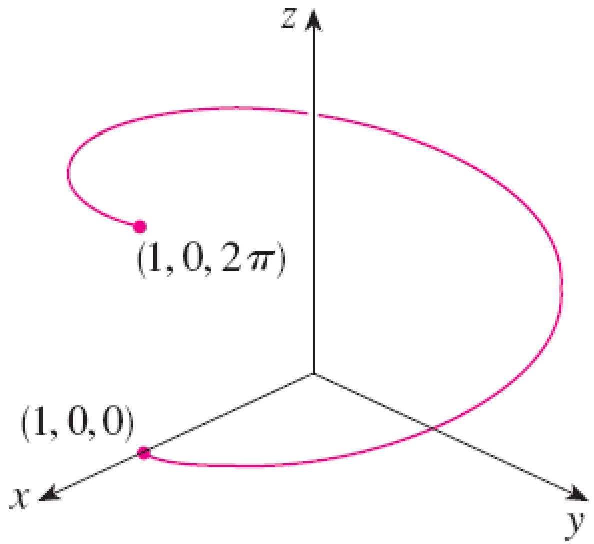
$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Example 1

- Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.



Parameterization

- A single curve C can be represented by more than one vector function.
- For example,

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad 1 \leqslant t \leqslant 2$$

can also be represented by the function

$$\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \qquad 0 \leqslant u \leqslant \ln 2$$

where the connection between the parameters t and u is given by $t = e^u$.

- We say that these two equations are *parameterizations* of the curve C.

Arc length function

- Suppose that C is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad a \leqslant t \leqslant b$$

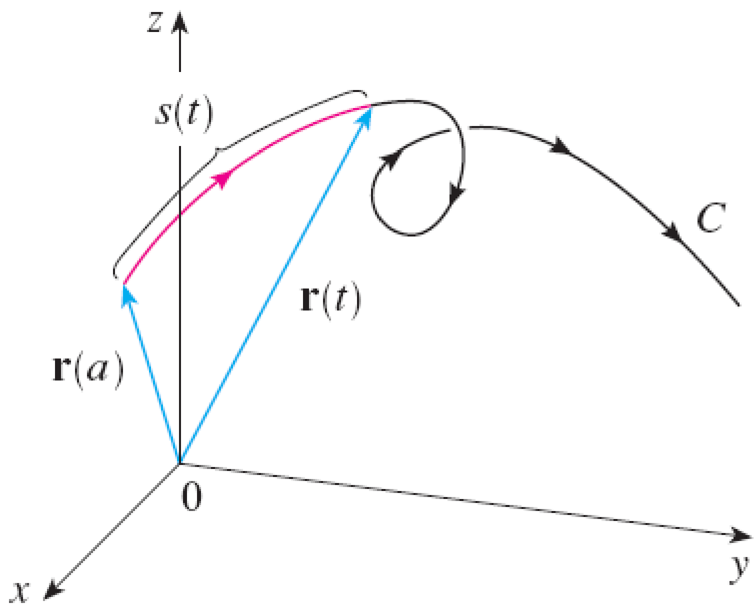
where $\mathbf{r}'(t)$ is continuous and traversed exactly once as t increases from a to b.
- We define its *arc length function* s by

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du$$

Arc length function

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

- Thus *s(t)* is the length of the part of *C* between $\mathbf{r}(a)$ and $\mathbf{r}(t)$.



Arc length function

- If we differentiate both side of the equation

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

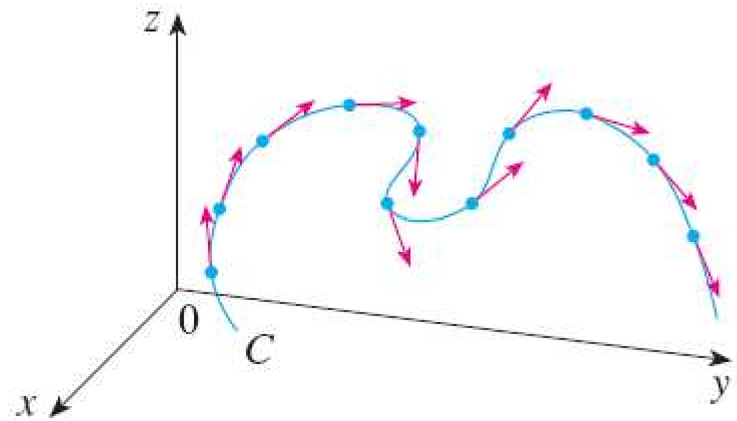
Parameterize a curve with respect to arc length

- It is often useful to parameterize a curve with respect to arc length because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system.
- If a curve $\mathbf{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function, then we may be able to solve for t as a function of $t = t(s)$.
- Then the curve can be reparameterized in terms of s by substituting for t: $\mathbf{r} = \mathbf{r}(\ t(s)\ )$.

Example 2

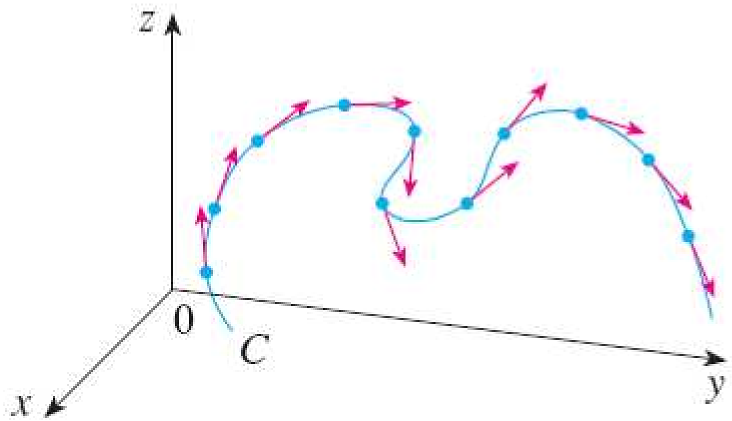
- Reparameterize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to arc length from $(1, 0, 0)$ in the direction of increasing t.

Curvature



- A parameterization $\mathbf{r}(t)$ is called *smooth* on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$ on I.
- A curve is called smooth if it has a *smooth* parameterization.
- A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

Curvature

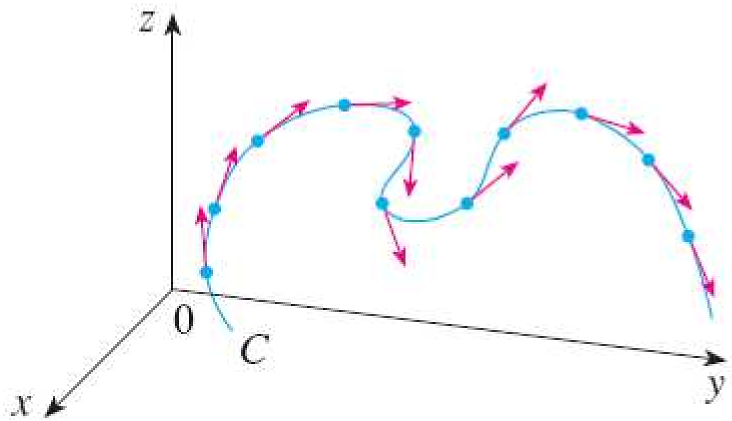


- If C is a smooth curve defined by the vector function $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

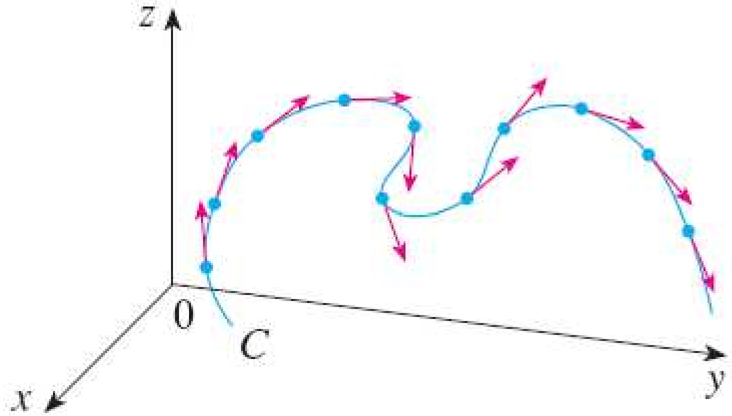
and indicates the direction of the curve.

Curvature



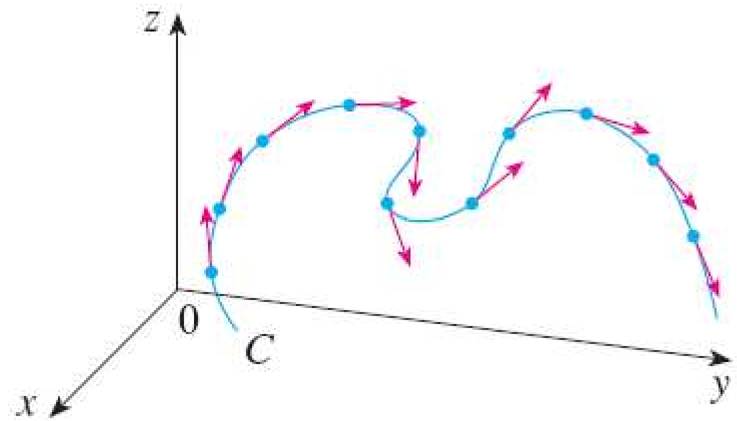
- From the figure, you can see that $\mathbf{T}(t)$ changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.
- The *curvature* of C at a given point is a measure of how quickly the curve changes direction at the point.

Curvature



- Specifically, we define the *curvature* to be *the magnitude of the rate of change of the unit tangent vector with respect to arc length*.
  - Note: we use arc length so that the curvature will be independent of the parameterization

Curvature



8 DEFINITION The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

- We can use the Chain rule to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \qquad \text{and} \qquad \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right|$$

so

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Curvature

10 THEOREM The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

PROOF Since $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ and $|\mathbf{r}'| = ds/dt$, we have

$$\mathbf{r}' = |\mathbf{r}'|\mathbf{T} = \frac{ds}{dt}\mathbf{T}$$

so the Product Rule (Theorem 13.2.3, Formula 3) gives

$$\mathbf{r}'' = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'$$

Using the fact that $\mathbf{T} \times \mathbf{T} = \mathbf{0}$ (see Example 2 in Section 12.4), we have

$$\mathbf{r}' \times \mathbf{r}'' = \left(\frac{ds}{dt}\right)^2 (\mathbf{T} \times \mathbf{T}')$$

Curvature

10 THEOREM The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Now $|\mathbf{T}(t)| = 1$ for all t, so $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal by Example 4 in Section 13.2. Therefore, by Theorem 12.4.6,

$$|\mathbf{r}' \times \mathbf{r}''| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T} \times \mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}||\mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'|$$

Thus
$$|\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{(ds/dt)^2} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}$$

and
$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$

□

Example 3

- Show that the curvature of a circle of radius a is $1/a$.

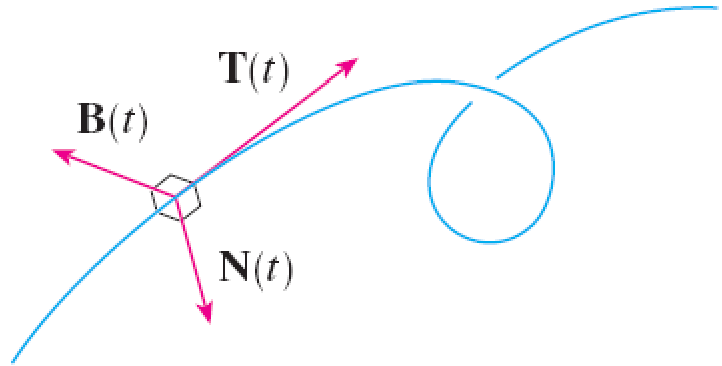
Example 4

- Find the curvature of the twisted cubic $\mathbf{r}(t) = <t, t^2, t^3>$ at a general point and at $(0, 0, 0)$.

Example 5

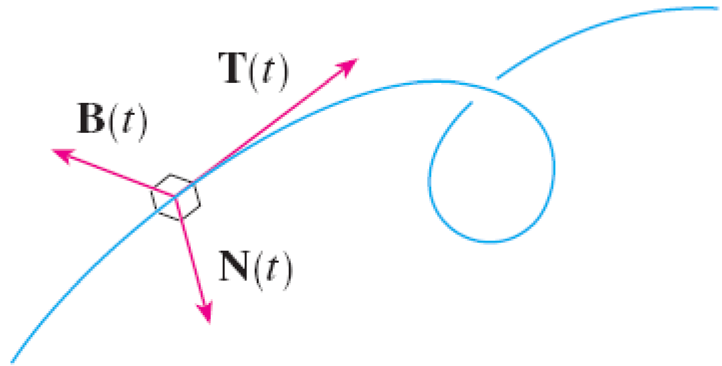
- Find the curvature of the parabola $y = x^2$ at the points $(0, 0)$, $(1, 1)$ and $(2, 4)$.

The normal and binormal vectors



- At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$.
- Because $\mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$.
  - Note that $\mathbf{T}'(t)$ itself is not a unit vector.

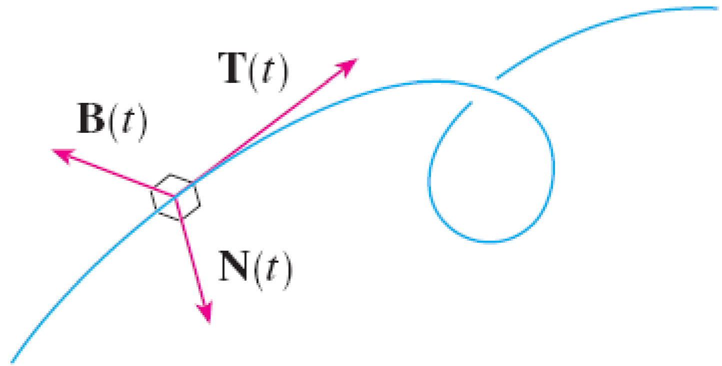
The normal and binormal vectors



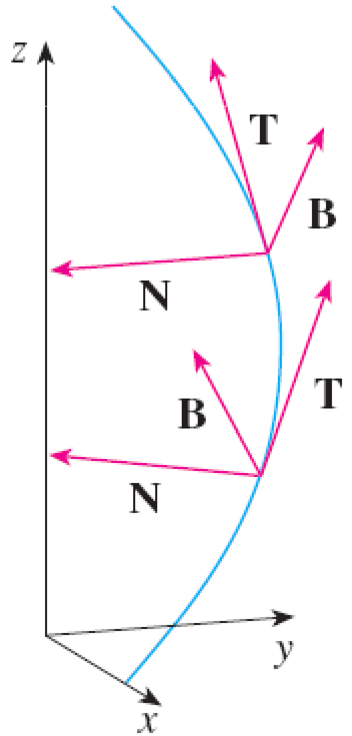
- If $\mathbf{r}'$ is also smooth, we can define the principle unit *normal vector* $\mathbf{N}(t)$ (or simply unit normal) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

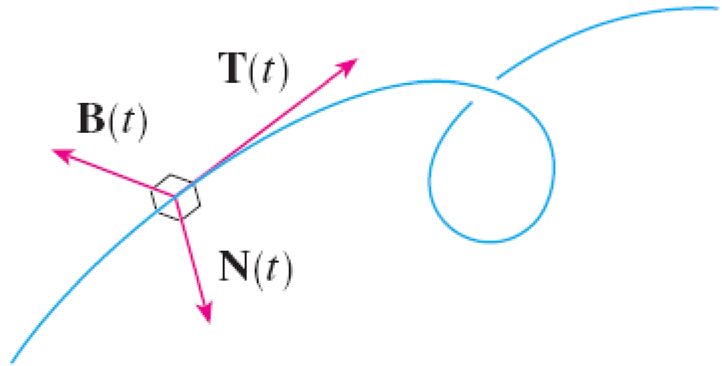
The normal and binormal vectors



- The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the *binormal vector*.
- It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector.



The normal and binormal vectors



- The plane determined by $\mathbf{N}$ and $\mathbf{B}$ at a point P on a curve C is called the *normal plane* of C at P.
- This plane consists of all lines that are orthogonal to the tangent vector $\mathbf{T}$.

Example 6

- Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

Example 7

- Find the equations of the normal plane of the helix in Example 6 at the point $P(0, 1, \pi/2)$.

MOTION IN SPACE: VELOCITY AND ACCELERATION

Velocity and acceleration

- In this section, we show how the ideas of tangent and normal vectors and curvature can be used in *physics* to study the motion of an object.

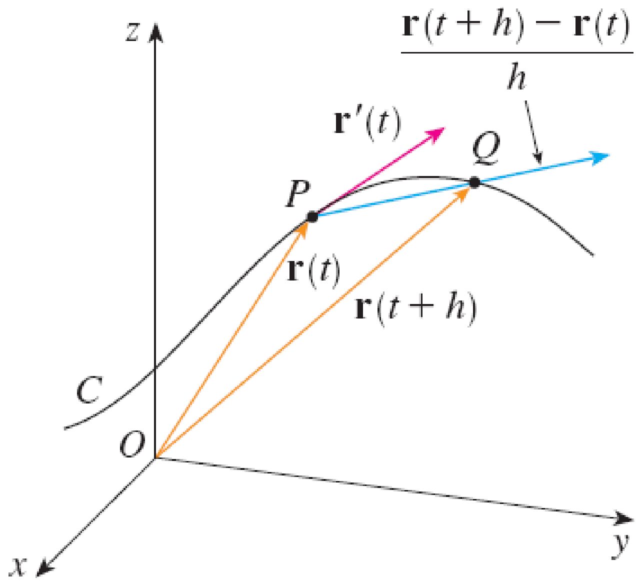
- Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$.

Velocity and acceleration

- For small value of h, the vector

$$\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$.



Velocity and acceleration

- The magnitude of $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ measures the size of the displacement vector per unit time.
- The vector gives the average velocity over a time interval of length h and its limit is the *velocity vector* $\mathbf{v}(t)$ at time t.

$$\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

- Thus the velocity vector is the tangent vector.

Velocity and acceleration

- The *speed* of the particle at time t is the magnitude of the velocity vector.

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt}$$

- So the speed is the rate of change of distance with respect to time.
- The *acceleration* of the particle is defined as the derivative of the velocity:

$$\boxed{\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)}$$

Example 1

- The position vector of an object moving in a plane is give by $\mathbf{r}(t) = t3\mathbf{i} + t2\mathbf{j}$. Find its velocity, speed, and acceleration when $t = 1$ and sketch them.

Example 2

- Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.

Example 3

- A moving particle starts at an initial point $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

Example 4

- An object with mass m that moves in a circular path with constant angular speed ω has position vector $\mathbf{r}(t) = a \cos \omega t\, \mathbf{i} + a \sin \omega t\, \mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin.